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Original Research

Elementary Collective Effects in Systems Containing Small Fermion-Numbers

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Abstract

Collective effects are a typical nuclear feature that refers to the behavior of a group of N fermions (protons and neutrons) within the atomic nucleus. Our interest lies in light nuclei only. Thus, we consider here, using the Lipkin model, a small-N fermion system at low-temperature T and discover -collective phenomena. Our fermion-model simplicity allows one to gain insight into collective fermion behavior. We focus attention, for $N < 20$, on several quantifiers. These include standard ones related to thermal behavior, such as entropy, and quantifiers of another kind, like quantum purity.

Keywords

Fermion systems; quantum mixture; finite temperatures; emerging collective effects

1. Introduction

The emergence of collective effects in a system with just a few fermions, like atomic nuclei, can be highly relevant and scientifically significant. These collective effects (like nuclear rotation, nuclear vibration, or nuclear deformation, for instance) refer to behaviors or phenomena that arise from the interactions between individual particles in a system. In the context of fermions, these interactions often involve quantum mechanical principles, such as the Pauli exclusion principle, which is a fundamental aspect of fermionic behavior. Observing collective effects in a small fermionic system can provide valuable insights into the underlying quantum behavior of particles. It can help researchers better understand how quantum statistics, particle interactions, and confinement in small systems lead us to discover emergent phenomena here. Note that we will not need this discovery to appeal to the entanglement notion, although this may be necessary to explain them. In condensed matter physics, understanding the behavior of a small number of fermions is crucial for comprehending electron correlations and interactions in specific materials, such as quantum dots, where a limited number of electrons are confined to a tiny space. As technology and experimental techniques advance, studying small fermionic systems can yield valuable contributions to understanding fundamental physics and developing new technologies.

The quantum N-fermion system exhibits some intricate properties [1-30]. Using statistical mechanics, we will study special manifestations of collective quantum properties for small ($N < 20$) fermion numbers at very low finite temperatures. We use a tractable many-body system, originally designed to study atomic nuclei, that can illuminate some interesting theoretical effects. Thermal statistical manipulation of the behavior of many fermions at finite temperatures is known to yield interesting insights [17]. Accordingly, we appeal to the exceedingly well-known Lipkin model (LM) [31, 32] at finite temperature and consider the pertinent structural traits in Gibbs' canonical ensemble formalism framework. The general principles of many-body physics and collective behavior that the Likpkin model addresses can be relevant to researchers in many areas.

1.1 A Nuclear Physics Motivation

Collective behavior is not limited to heavy atomic nuclei; even light atomic nuclei can exhibit collective effects that arise in nuclei due to the interactions among nucleons (protons and neutrons). While the specific manifestations of collectivity may vary depending on the size and characteristics of the nucleus, certain light nuclei do display collective behavior. For example, some light nuclei exhibit clustering behavior, where nucleons arrange themselves into clusters resembling alpha particles (helium nuclei). Enhanced stability can result from this clustering phenomenon observed in certain isotopes. Other collective effects are nuclear deformations. Also, light nuclei can undergo collective excitations, where nucleons collectively oscillate or move in a coordinated manner. This can include vibrations and rotations [10].

1.2 Our Goal

For small particle numbers $N < 20$, we focused on emerging collective phenomena in a celebrated interacting fermions model. These phenomena appear in various fields of science, particularly nuclear physics, and give rise to typical behaviors revealing a high level of organization.

Our examples highlight how, for small particle numbers, collective effects in fermion systems can manifest themselves. LARGE particle numbers emerge in various physical phenomena, from superconductivity and metal-insulator transitions to quantum (and fractional) Hall effect.

2. Details of the Lipkin Model to Be Needed Here

2.1 Preliminaries

Many-body theorists are usually knowledgeable about exactly solvable models (ESM). There are not many of them. They help gain insight into many particle systems. If the problem to be solved can be related to a precisely solvable one, however vaguely, one can usually gain some insight. In this work, we use a model that, with a little diagonalization effort, yields exact results.

The Lipkin Model (LM) [31, 32] is a simplified quantum mechanical model, easily diagonalizable, used to describe certain features of many body systems. Harry J. Lipkin introduced it in 1960; since then, it has been a valuable tool for understanding collective phenomena. This model is primarily used to study the interplay between single-particle motion (which can be described by a mean-field approach) and residual two-body interactions (correlations between fermions pairs) in a many-body quantum system. The LM provides a simplified framework to explore distinct many-fermion traits. Next, some of them are mentioned [31, 32].

1) One of the key features of the Lipkin Model is its ability to describe pairing correlations. This pairing phenomenon is essential for understanding superfluidity and superconductivity.
2) Phase Transitions: the model can be used to study phase transitions in many body systems, such as the transition from a non-correlated phase to a correlated phase.
3) Symmetry Breaking: the Lipkin Model can illustrate the breaking of certain symmetries. In particular, it can show how the pairing interaction can break isospin symmetry in certain cases.
4) Didactics: the Lipkin Model is often used as a pedagogical tool because it provides a tractable way to explore many-body interactions and their effects.

The LM itself [3] involves a set of quantum operators and Hamiltonians that describe the interaction between nucleons and is typically solved numerically.

In summary, models like Lipkin's are indispensable tools in understanding and exploring quantum phenomena, testing theoretical methods, and providing insights into the behavior of complex quantum systems. Their importance extends beyond theoretical physics and has applications in condensed matter physics, quantum information, nuclear physics, and related fields [31, 32].

2.2 Lipkin Model's Formalism

The LM [31, 32] is a simplified system containing N fermions in just two levels. It is exactly solvable. The model considers a quite simple fermion-fermion interaction of strength v. In nature, of course, the coupling constants are fixed. In the model, of course, we vary it to observe how changes in v affect the ground state traits. We also study the model behavior for different N.

Our model possesses $N = 2\Omega$ fermions that occupy two different N-fold degenerate single-particle (sp) energy levels. A sp energy gap ϵ.characterizes them. This entails 4Ω s.p. microstates. Two quantum numbers ($\mu = \pm 1$ and $p = 1,2, \dots, N$) are associated with a given microstate $|p, \mu >$. The first one, called μ, adopts the values $\mu = -1$ (lower level) and $\mu = +1$ (upper level). The second runs from unity to N. This remaining quantum number, called p, is baptized as a quasi-spin or pseudo-spin that singles out a specific microstate about the $2N$-fold degeneracy. The pair p, μ is considered a "site" that can either be occupied by a fermion or remain empty. Lipkin fixes [31, 32].

$$N = 2J. \tag{1}$$

Here, J is a sort of angular momentum. Lipkin [17, 31] uses special operators called quasi-spin ones. We use below the usual creation operators $C^{+}_{p,\mu}$ and its associated destruction ones $C_{p,\mu}$ for creating or destroying a fermion at a site $|p, \mu>$.

2.3 Quasi Spin Operators

Quasi-spin operators J are mathematical constructs describing the collective properties of specific many-body systems. These operators arise in various areas of physics, such as nuclear physics, condensed matter physics, and quantum optics, where systems can exhibit collective behavior due to interactions between constituent particles. Quasi-spin operators are beneficial in cases where the collective behavior resembles the behavior of spin systems, hence the name "quasi-spin." The concept of quasi-spin originates from the analogy between the properties of many-body systems and those of spin systems, which are well-understood and widely used in quantum mechanics. In a spin system, the angular momentum operators (spin operators) obey the commutation relations of the SU2 algebra, and they play a fundamental role in characterizing the system's angular momentum and magnetic properties. In many-body systems, introducing quasi-spin operators represents collective excitations or modes that behave similarly to angular momentum. These operators often have algebraic properties resembling the SU2 algebra, making them suitable for describing the collective dynamics of the system. Overall, quasi-spin operators offer a valuable tool in theoretical physics for investigating collective behavior in complex many-body systems, facilitating the understanding of emergent phenomena and enabling the development of analytical and numerical techniques to study these systems in different physical contexts [31, 32]. The specific form and properties of the quasi-spin operators depend on the nature of the many-body system being studied and the interactions between its constituents. They are introduced to simplify the description of collective phenomena and, as mentioned earlier, provide a robust mathematical framework for treating many interacting fermions. One has for these operators the definitions:

$$J_z = \sum_{p,\mu} \mu \, C^{+}_{p,\mu} C_{p,\mu}, \tag{2}$$

$$J_+ = \sum_{p} C^{+}_{p,+} \, C_{p,-}, \tag{3}$$

$$J_- = \sum_{p} C^{+}_{p,-} \, C_{p,+}, \tag{4}$$

and the Casimir operator:

$$J^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+). \tag{5}$$

The eigenvalues of J^2 take form $J(J+1)$, and the Lipkin Hamiltonian reads (v is a coupling constant) [31, 32]:

$$H = \epsilon J_z + \frac{v}{4}(J_+^2 + J_-^2). \tag{6}$$

The Hamiltonian matrix is if n denotes the number of fermions in the upper level [31, 32],

$$\begin{aligned}\langle n'|H_L|n\rangle = &\left\{\frac{N}{2} - n + 1\right\}\delta_{n',n} - \\ &-\frac{v}{2}\sqrt{(N-n)(N-n+1)(n+1)n}\,\delta_{n',n+2} \\ &-\frac{v}{2}\sqrt{(N-n)(N-n+1)(n+1)n}\,\delta_{n',n-2},\end{aligned} \tag{7}$$

with $n = 0,1,\ldots,N$ for $J = N/2$. Numerical diagonalization yields energy-eigenvalues $E_n(v,J)$. These eigenvalues are needed to build the partition function Z (see below).

3. Detecting Collective Effects in the Associated Spectrum

We now enter new territories and begin to address our goal of detecting collective effects. For this, let us consider the energy difference between the ground state energy E_0 and the first excited one E_1 and focus on the energy gap $\Delta E = E_1 - E_0$.

We plot ΔE versus the coupling constant v in Figure 1. For $N = 2$ the energy gap monotonously grows with the strength v. For more significant fermion numbers, and things drastically change—a critical value for v energy with a minimal energy gap. The gap is no longer a monotonously growing one. We face a collective effect here.

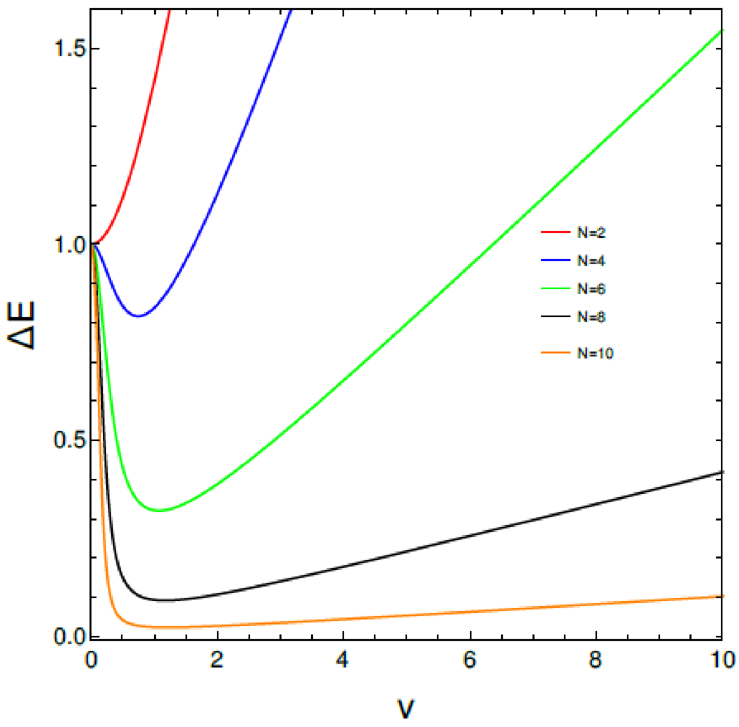

Figure 1 Energy gap $\Delta E = E_1 - E_0$ between the first excited and ground state energies. The gap is plotted about the coupling constant v. We see that a collective effect emerges at $N = 4$. Instead of growing monotonously with the coupling strength, the gap first diminishes, reaches a minimum, and then starts to grow again. The minimum is a collective effect that requires at least four fermions to be produced.

4. Thermal Collective Effects

Now, we will try to detect collective effects only seen at finite temperature T. As T is tiny, our putative effects will resemble ground-state workings. In the following expressions, we refer to the inverse temperature β and to entropy S. We need Gibbs' canonical framework here to proceed. The procedure is detailed in [33]. All thermal quantities of interest are derived from the partition function Z [4]. We construct Z using probabilities assigned to the models' microscopic states using the Lipkin energies E_i [4]. Significant macroscopic quantifiers are computed following the procedure in [4]. These indicators and Z derive from the canonical probability distributions [4] $P_n(v, J, \beta)$. The relevant expressions are provided in [4]. Below are the appropriate expressions for constructing the required apparatus.

$$P_n(v, J, \beta) = \frac{1}{Z(v, J, \beta)} e^{-\beta E_n(v,J)} \tag{8}$$

$$Z(v, J, \beta) = \sum_{n=0}^{N} \mathrm{e}^{-\beta E_n(v,J)} \tag{9}$$

$$\begin{aligned} U(v, J, \beta) = \langle E \rangle = -\frac{\partial lnZ(v, J, \beta)}{\partial \beta} = \\ = \sum_{n=0}^{N} E_n(v, J) P_n(v, J, \beta) = \\ = \frac{1}{Z(v, J, \beta)} \sum_{n=0}^{N} E_n(v, J) \mathrm{e}^{-\beta E_n(v,J)} \end{aligned} \tag{10}$$

$$S(v, J, \beta) = -\sum_{n=0}^{N} P_n(v, J, \beta) \ln\left[P_n(v, J, \beta)\right] \tag{11}$$

Thermal quantifiers provide much more information than the one obtained via just the quantum resources of zero temperature T [4]. Taking a low enough T, thermal quantifiers yield an excellent representation of the $T = 0$ scenario [4]. We plot the entropy S versus N in Figure 2. S almost vanishes for just two fermions and starts growing as the number N grows beyond two. We use the inverse temperatures $\beta = 5$ and 20 and a strength $v = 1$. The emerging collective phenomenon here is the growth of the entropy as N augments. Another interesting effect is S- saturation at the exact value of N for two different βs. In this, we are observing an N effect and not a T one.

We pass now to the mean energy U, another thermal quantifier. Figure 3 depicts the mean energy U versus N for $v = 1$ and several β values. One interprets the growth of the binding energy with N as a collective effect.

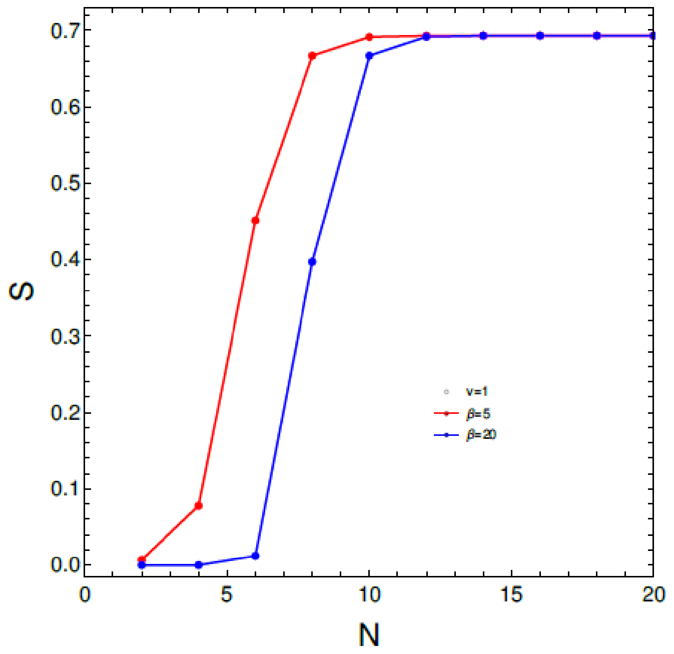

Figure 2 Lipkin entropy S versus particle number for two inverse temperatures and $v = 1$. The entropy almost vanishes at $N = 2$ and significantly augments for larger N values. Another collective phenomenon is detected as well. There is a saturation S-effect for relatively low N values.

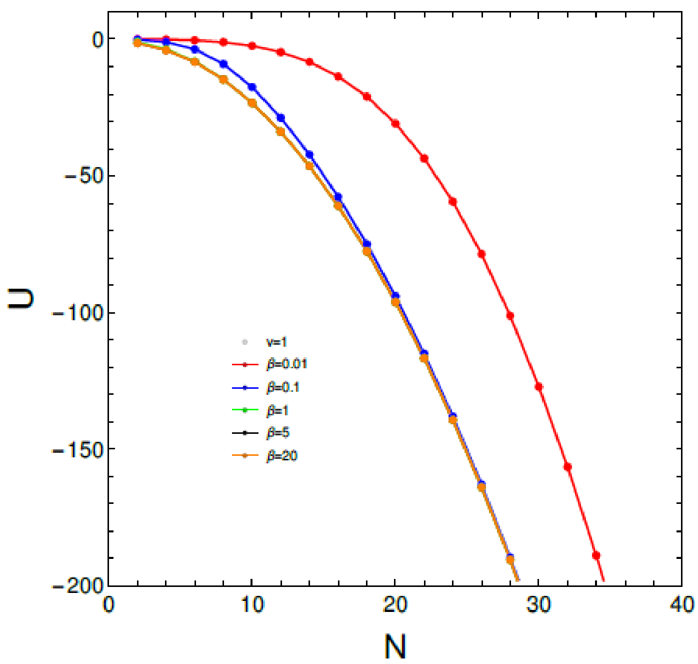

Figure 3 Lipkin mean energy U (binding energy) versus particle number for several inverse temperatures and $v = 1$. This simple plot shows that the mean energy vanishes

at $N = 2$ and rapidly diminishes or larger N values, clearly another collective phenomenon.

5. A State's ρ Degree of Mixture C_f

5.1 Preliminaries

As stated above, in quantum mechanics, the notion of purity P_y is a measure of how "mixed" or "pure" a quantum state (QS) is [30]. When a single quantum wavefunction describes a QS, this represents an instance of complete knowledge about the QS. On the other hand, a mixed state is a statistical set of multiple pure states. Thus, it represents a state of incomplete knowledge or uncertainty about the system. We repeat thus that in quantum mechanics, quantum states can exist in two fundamental distinct forms: pure and mixed. A pure state that a single, normalized wave function can describe exhibits maximal coherence and well-defined quantum properties.

On the other hand, a mixed state is a statistical ensemble of pure states $|i>$, each with its associated probability p_i. It exhibits less coherence and may have probabilistic uncertainties. The degree of mixing or superposition in a quantum state is measured here by the quantum mixing quantifier C_f. The purity P_y of a quantum state quantifies its coherence and measures its proximity to being pure. It is the trace of the square of the state's density matrix ρ as $P_y = Tr(\rho^2)$. For a pure state, the purity equals 1, while for a mixed state, the purity is less than 1 [34].

The degree of quantum mixture C_f equals unity less than the quantum purity P_y [30, 34].

$$C_f = 1 - Tr(\rho^2) = 1 - \sum_i p_i^2 \,. \tag{12}$$

Note that we have $C_f = 0$ and $P_y = 1$ for pure states. C_f is a significant quantity for us here. In probability terms, one has $P_y = \sum_{n=0}^{N} \big(P_n(v, J, \beta)\big)^2$ and $C_f = 1 - P_y^2$.

5.2 Results for C_f

Figure 4 depicts C_f versus N for several strengths v and $\beta = 10$. We see that $v = 0$, the unperturbed system's state is pure for all N and thus C_f vanishes. If $v \neq 0$, the state remains pure if $N = 2$ but grows for more significant particle numbers and eventually saturates for N large enough.

A similar collective effect is seen in Figure 5 for $N = 4$ and not for $N = 2$ by plotting C_f versus v for two different β values.

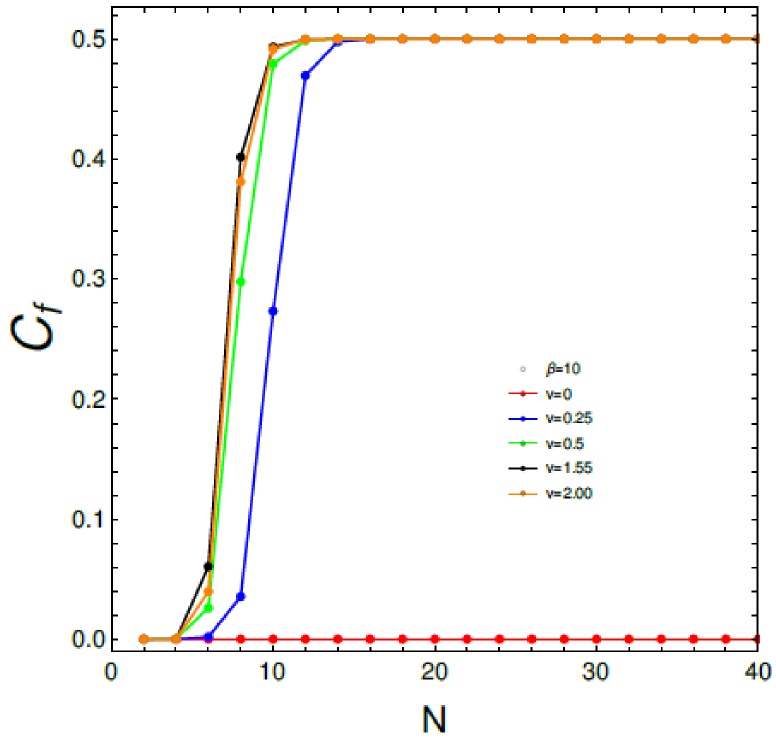

Figure 4 Mixing degree C_f versus particle number N for variegated strengths v. We detect sudden jumps in C_f for critical $N = 4$ or larger particle number values. These jumps are our collective effect.

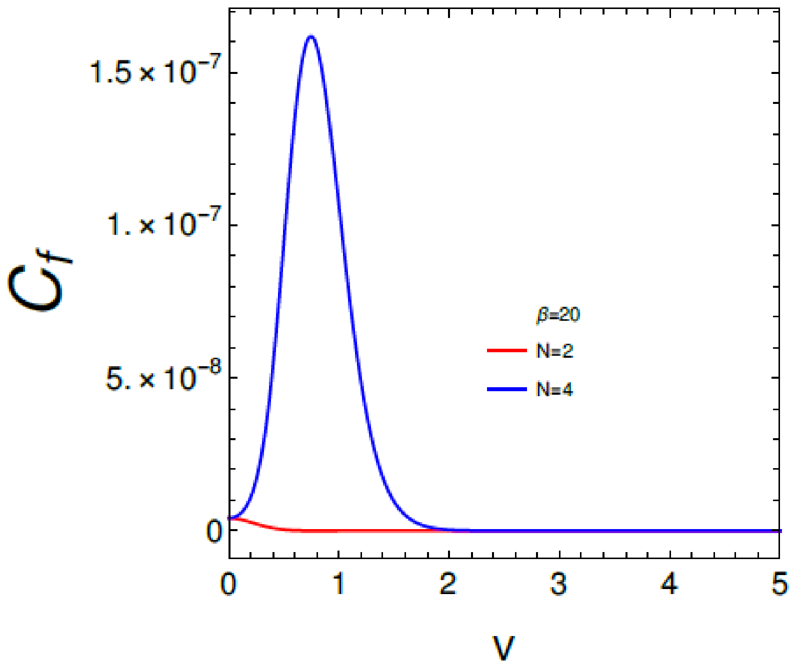

Figure 5 Mixing degree C_f versus particle numbers $N = 2$ and $N = 4$ versus the strength v for $\beta = 20$. We detect a clear collective jump in C_f for $N = 4$. This jump is our collective effect.

For $N = 2$, the disappearance of the degree of mixture (purity) might indicate that the two fermions form a highly correlated or entangled pair. In a two-body system, the ground state can be pure, meaning no mixture or uncertainty is associated with the state. For $N = 4$ to $N = 10$, we find that the growth of the degree of mixture suggests an increase in the complexity of the quantum correlations within the system. With more fermions, interactions between particles become more significant, resulting in a more intricate interplay of quantum states. The system may become more entangled, and correlations among fermions may increase, resulting in a less pure state and a higher degree of mixture. The saturation of the degree of mixture around 0.5 for larger values of N is an interesting observation. This saturation might indicate that a certain level of quantum chaos or complexity has been reached, beyond which additional particles do not significantly alter the overall degree of mixture. The system may have reached a quantum-thermal equilibrium state where further particles do not introduce new correlations or dramatically affect the quantum coherence.

6. Conclusions

Many emerging collective many-body effects (ECMBE) are observed in condensed matter physics, nuclear physics, and various other areas of physics. Understanding and characterizing these phenomena often require advanced theoretical and experimental techniques, and they are essential for explaining the behavior of complex systems at the quantum level.

Here, we have used the Lipin model to show that already at its elementary level of complexity, these ECMBE are detected employing relatively unsophisticated may body techniques.

Thus, a collection of a few interacting fermions tends to behave dramatically differently than those displayed by just two fermions. We emphasize the vital role played by the $N = 4$ instance. This would correspond to the alpha particle, one of the most stable nuclei in the periodic table.

Author Contributions

Investigation, D. M., A.P. and A.R.P.; Project administration, A.P.; Writing—original draft, D.M. A.P., and A.R.P. All authors have read and agreed to the published version of the manuscript.

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Competing Interests

The authors declare no conflict of interest.

Additional Materials

The following additional materials are uploaded at the page of this paper.

1. Appendix.

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