

Research Article

Development of the Stochastic Interpretation of Quantum Mechanics by E. Nelson. Derivation of the Schrödinger-Euler-Poisson Equations

Mikhail Batanov-Gaukhman *

Institute No.2 “Aircraft, rocket engines and power plants”, Federal State Budgetary Educational Institution of Higher Education “Moscow Aviation Institute (National Research University)”, Volokolamsk highway 4, Moscow, Russian Federation; E-Mail: alsignat@yandex.ru

* **Correspondence:** Mikhail Batanov-Gaukhman; E-Mail: alsignat@yandex.ru

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Abstract

The article aims to develop the stochastic interpretation of quantum mechanics by E. Nelson based on balancing the intra-systemic contradiction (i.e., antisymmetry) between “order” and “chaos”. For the set task, it is proposed to combine two mutually opposite system-forming principles: “the principle of least action” and “the principle of maximum entropy” into one, the “principle of averaged efficiency extremum”. In a detailed consideration of the averaged states of a chaotically wandering particle, the time-independent (stationary) and time-dependent stochastic Schrödinger-Euler-Poisson equations are obtained as conditions for finding the extremals of the globally averaged efficiency functional of the stochastic system under study. The resulting stochastic equations coincide with the corresponding Schrödinger equations up to coefficients. In this case, the ratio of the reduced Planck constant to the particle mass is expressed through the averaged characteristics of a three-dimensional random process in which the considered wandering particle participates. The obtained stochastic equations are suitable for describing the quantum states of stochastic systems of any scale.



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Keywords

Derivation of the Schrödinger equation; stochastic quantum mechanics; stochastic equation; order and chaos; intra-system antisymmetry

1. Background and Introduction

Several dozen interpretations of Quantum Mechanics (QM) exist in modern physics. Each has its advantages and disadvantages. Still, none is precisely defined since many researchers often put different meanings into the same concepts.

One of the reasons for this situation in quantum physics is associated with a different attitude to the wave function $\Psi(x, t)$.

Most experts agree with M. Born's statement that the square of the modulus of the wave function of a particle $\Psi(x, t)$ is equal to the probability density function (PDF) of the particle's location at a point x

$$|\Psi(x, t)|^2 = \rho(x, t).$$

However, it should be borne in mind that, in general, this PDF is composed of several factors associated with the measurement process.

$$|\Psi(x, t)|^2 = \rho(x, t) = f[\rho_p(x, t), \rho_m(x, t), \rho_e(x, t), \rho_d(x, t), \rho_o(x, t)], \quad (0.1)$$

where

$\rho_p(x, t)$ is the PDF associated with the chaotic behavior of the particle;

$\rho_m(x, t)$ is the PDF associated with the method errors;

$\rho_e(x, t)$ is the PDF associated with the influence of the external environment;

$\rho_d(x, t)$ is the PDF associated with the instrument errors;

$\rho_o(x, t)$ is the PDF associated with the operator errors.

An example of functional dependence (0.1) is the PDF.

$$|\Psi(x, t)|^2 = \rho(x, t) = \frac{1}{\sqrt{2\pi[\sigma_{px}^2 + \sigma_{mx}^2 + \sigma_{ex}^2 + \sigma_{dx}^2 + \sigma_{ox}^2]}} \exp\left\{-\frac{x^2}{2[\sigma_{px}^2 + \sigma_{mx}^2 + \sigma_{ex}^2 + \sigma_{dx}^2 + \sigma_{ox}^2]}\right\}, \quad (0.2)$$

where

$$\sigma_{ix}^2 = \sigma_{ix}^2(t) = \int_{-\infty}^{+\infty} \rho_i(x, t)x^2 dx \quad (\text{here } i = p, m, d, o),$$

is a variance of the i -th influencing factor on the X axis measurement result.

All of the above factors are present when measuring the physical characteristics of particles of any scale. However, these factors affect the result differently depending on the particle size.

At the same time, almost all specialists who study the properties of non-relativistic pico-particles (i.e., particles with characteristic sizes of atomic and subatomic scales, $10^{-8} - 10^{-13}$ cm) use the same

mathematical apparatus of quantum mechanics (QM), designed to predict possible configurations and evolutions of wave functions $\Psi(x, t)$ of one particle or an ensemble of identical particles.

Focusing on certain factors influencing the measurement process using the same mathematical apparatus leads to different interpretations of QM.

For example, in several experiments, pico-particles are extremely sensitive to the influence of the measuring system and the observer on them. The effects associated with the fact that the operator, by his presence, introduces parasitic capacitance and inductance into radio-electronic devices, which leads to an imbalance of electronic resonators, etc., are widely known.

In this case, the wave function (0.1) should consider all influencing factors. At the same time, the methodology for perceiving the results obtained is most consistent with the Copenhagen interpretation of QM, formulated by Niels Bohr and Werner Heisenberg.

In other experiments, the factors interfering with the measurement are so insignificant that they can be neglected. For example, we judge the possible states of an electron in a hydrogen atom by its emission spectrum. If we abstract from the slight broadening of the spectral lines associated with the influence of various accompanying factors, then in this case PDF (0.1) takes the form

$$|\Psi(x, t)|^2 = \rho(x, t) = f[\rho_p(x, t)\rho_e(x, t)].$$

This wave function characterizes only the properties of the electron itself, taking into account the influence of the vacuum, leading to the Lamb shift of the spectral lines.

In this article, we will adhere to the Stochastic interpretation of quantum mechanics, most clearly expressed in the works of Edward Nelson [1-3], published in 1966-1985.

In addition to E. Nelson, this interpretation of QM was developed by R. Fürth [4], I. Fényes [5], W. Weizel [6], M. Pavon [7], K. Namsrai [8]. An alternative stochastic interpretation of QM was designed by R. Tsekov [9, 10]. S. Milz & K. Modi [11]

Among the more recent works on stochastic quantum mechanics, the articles by J. Lindgren & J. Liukkonen (2019) [12] and T. Koide & T. Kodama (2018) [13], F. Kuipers (2023) [14] should be noted.

Nelson's stochastic interpretation is associated with the logical construction of QM by analogy with the theory of Brownian motion [more precisely, the Ornstein-Uhlenbeck process].

In Nelson's interpretation, the reason for the chaotic behavior of a pico-particle is associated with the effect of vacuum fluctuations on it. The diffusion coefficient of such a stochastic process turns out to be imaginary due to the absence of friction and the specificity of the vacuum viscosity. Thus, in the stochastic interpretation of QM, the primary is not the wave function $\Psi(x, t)$ but complex small-scale curvatures of the space-time continuum (i.e., the Wheeler-Bohm-Vigier "quantum foam"), which affect to the colloidal pico-particle.

In this case, PDF (0.1) takes the most straightforward form

$$|\Psi(x, t)|^2 = \rho(x, t) = \rho_e(x, t) = \psi(x, t)\psi^*(x, t), \quad (0.3)$$

since it characterizes only the chaotic behavior of a structureless particle under the influence of a turbulent environment.

Recall that the Langevin and Fokker-Planck stochastic equations describe Brownian motion without considering the structure of particles and the uncertainty associated with measurement errors. However, a fundamental difference exists between Brownian particles ($\sim 10^{-4}$ cm) and pico-

particles ($\sim 10^{-8} - 10^{-13}$ cm). Brownian (colloidal) particles can be observed practically without affecting them with a microscope, while pico-particles, in principle, cannot be observed directly.

In this article, the maximally simplified (more precisely, not taking into account the measurement error and the influence of other particles) probability amplitude (PA) $\Psi(x, t) = \psi(x, t)$ will be called the “pure” wave function.

It should be noted that in most books and textbooks on quantum mechanics, initially, the PA $\Psi(x, t)$ means the “pure” wave function of the particle. This is one of the reasons for the lack of agreement between theorists, experimenters, and specialists working in various fields of quantum physics. It was the attitude to the “pure” or “impure” wave function $\Psi(x, t)$ that caused the disputes between A. Einstein (who studied Brownian motion in his youth) and N. Bohr (whose early works were associated with atomic emission spectra).

So, in this article, under the stochastic interpretation of quantum mechanics by Edward Nelson, we mean a version of QM in which the wave function $\psi(x, t)$ characterizes only the chaotic behavior of a wandering particle under the influence of environmental fluctuations, without taking into account measurement errors and the influence of the operator. This particle (like a Brownian corpuscle) has a volume and a chaotic motion trajectory. In this case, the wave function $\psi(x, t)$ has the statistical character attributed to it by M. Born.

At the same time, it is taken into account that within the framework of Nelson’s stochastic interpretation of the QM, the “pure” PA $\psi(x, t)$ turns out to be a kind of “intellectual thing-in-itself”. This is because the “pure” wave function $\psi(x, t)$ can be found only by solving stochastic differential equations. Any attempt to perform a measurement will lead to a partial distortion or destruction of the stochastic system under study and, hence, to a change in its PA $\psi(x, t)$.

This article attempts to develop the foundations of mass-independent stochastic quantum mechanics (MSQM), which is a development of Nelson's stochastic quantum mechanics (SQM) [1-3], and proposes a solution to the problem of measuring “pure” parameters of stochastic quantum systems.

The article attempts to exclude mass from all the physical quantities and constants mentioned here. This is one of the steps in the program of complete clearing of physics from the concept of “mass” and its dimension “kilogram” (or “pound”, etc.), since this phenomenological concept, in the author's opinion, is superfluous and hinders the development of fully geometrized physics.

A probabilistic model of a chaotically wandering particle (ChWP) is considered, which, like the pico-particle by E. Nelson [1], has a volume and a continuous trajectory of motion. However, compared to the SQM [2], the MSQM has no restrictions on the investigated particle size. Based on this model (by the calculus of variations method, different from [1]), stochastic Eq.s (37), (41), (67), (68) are derived, which are a generalization of the Schrödinger equations, and also was obtained Eq. (73) corresponding to the form of the diffusion equation (a particular case of the Fokker-Planck-Kolmogorov equation).

The author believes the main advantage of the obtained stochastic equations is that they are suitable for describing the averaged quantum states of chaotically wandering particles (ChWP) of any scale and quality. For example, this approach is ideal for describing quantum effects when averaging the chaotic displacements of the geometric (or hypothetical) center: the tip of a tree branch under the influence of wind blows or random displacements of the center of the planet's core, or changes in stock prices on the stock exchange, or displacement concentration of the student's attention from the subject of study, etc.

In recent years, many advances have been made in analyzing stochastic nonlinear Schrödinger equations. In paper [15], the authors discovered a symplectic structure on the Wasserstein density manifold of the stochastic nonlinear Schrödinger equation. Graph-based discretization was proposed in [16] to preserve such a structure numerically. The stochastic dispersion relation and the associated optimal control problem were reported in [17]. More recently, in [18], the authors showed the correctness of stochastic logarithmic Schrödinger equations governed by general multiplicative noise.

The article is based on the assertion that the dichotomy “Order and chaos” affects any stochastic system (including a chaotically wandering particle of any scale, for example, a fish in a round aquarium, the center of mass of the planet’s core, the nucleus of a biological cell, a mosquito in a swarm of mosquitoes, electron in an atom, center of mass of the yolk in an egg, Brownian particle in a colloidal solution, etc.).

The concept of “order” is formalized through the “principle of least action”, and the idea of “chaos” is formalized through the “principle of maximum entropy”. The article proposes to combine these principles into one “principle of extremum of average efficiency.” This principle made it possible to write down the function of the globally averaged “efficiency” of a chaotically wandering particle. In this case, the extremal equations for this function in a particular case are the Schrödinger equations, in which the ratio of Planck’s constant to the particle mass (\hbar/m) is replaced by the ratio of the averaged parameters of the random process under study (standard deviation σ to the autocorrelation coefficient)

$$\frac{\hbar}{m} \approx \frac{2\sigma^2}{\tau_{acor}}$$

This makes it possible to use the obtained stochastic equations (including the Schrödinger equations) to study the averaged behavior of a chaotically wandering particle of any scale. For example, this approach is suitable for describing quantum effects when averaging chaotic movements of a geometric (or hypothetical) center: the tip of a tree branch under the influence of wind blows or random displacements of the center of the planet’s core, or fluctuations in the stock price on the stock exchange, or a shift in the student’s concentration from the subject of study and so on.

The main achievement of this article is the analytical derivation of the Schrödinger-Euler-Poisson equation, which is suitable for describing stochastic processes and quantum effects of any scale. In particular, under certain conditions, the Schrödinger-Euler-Poisson equation is transformed into the Schrödinger equations up to coefficients. It has not been possible to derive the Schrödinger equations practically since their appearance in 1925. At the same time, in deriving the Schrödinger-Euler-Poisson equation, it was possible to understand many features of random processes and quantum phenomena, both the microworld and the macroworld.

2. Method

Stochastic objects (systems) are formed while simultaneously striving for “order” and “chaos” [19, 20]. This is due to the influence of two mutually opposite system-forming factors on any stochastic object: the “principle of least action” and the “principle of maximum entropy”. Stochastic

systems' opposite tendency to “order” and “chaos” leads them to equilibrium, optimization, energy saving, energy efficiency, and space-time symmetry.

In this article, an attempt is made to combine the two above-mentioned mutually opposite (antisymmetric) principles into one, the “principle of averaged efficiency extremum” (PAEEx), which is internally balanced concerning “determinism” (i.e., predetermination) and “randomness”. This new principle theoretically substantiates the empirically manifested advantage of symmetric states of the free stochastic objects.

2.1 Probabilistic Model of a Wandering Particle

Consider a solid particle with volume and trajectory of motion. Suppose that under the influence of fluctuations of the environment, the particle constantly wanders chaotically in the vicinity of the conditional center, combined with the origin of the coordinate system XYZ (see Figure 1).

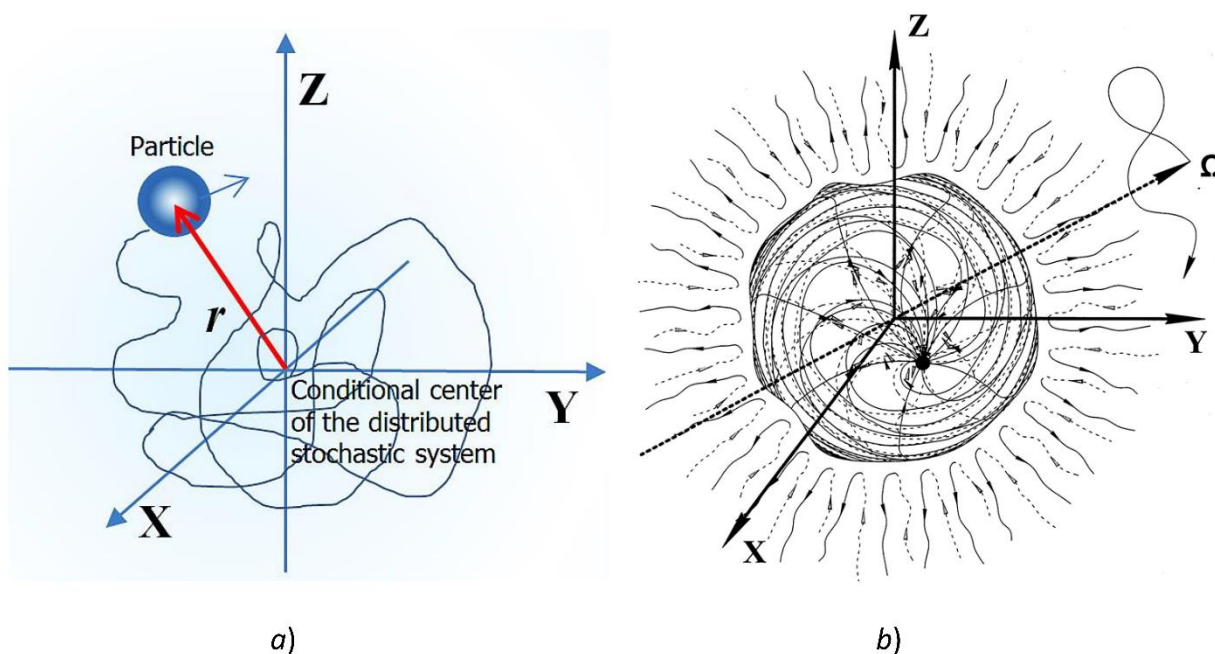


Figure 1 a) A simplified model of the solid particle (more precisely, the geometric or hypothetical center of the object) that wanders chaotically in the vicinity of the conditional center due to the random force influences of the fluctuating environment. b) Examples of a chaotically wandering particle (ChWP), whose deviation from the center creates environmental tensions.

Such a chaotically wandering particle (ChWP) can be attributed to random displacements in three spatial (or conditional, phase) measurements with time of **the geometric (or hypothetical) center** of the following objects or subjects: of a valence electron in a hydrogen-like atom, of an oscillating atom in a crystal lattice, of a trembling nucleus of a biological cell, of a fluttering heart in the chest of an animal, of a fluttering yolk in a chicken egg, of a fluttering moth in the vicinity of a burning lamp, of floating fish in the aquarium, of a shifting mosquito swarm, of a flying mosquito in a swarm, of a trembling organelle in the cytoplasm, of an oscillating biological cell in living tissue, of a vibrating incandescent core in the bowels of the planet, of a wandering pollen in diluted sugar syrup, of a

rushing air bubble in a boiling liquid, of a wiggling embryo in the womb, of a shifting school of fish in the ocean, of a moving astronaut in a space station module, of a shifting galaxy in outer space, of a fluttering flower in the wind, a fluctuating cost of service in the labor market, changes in public opinion regarding a social problem, distractions from a situation which a scientist is trying to solve, etc.

In other words, the random behavior of most of the 3-dimensional objects, subjects, and systems surrounding us in the presence of a geometric (or hypothetical) center and a sufficiently large observation period can be interpreted as the behavior of the chaotic wandering particle (ChWP) in a 3-dimensional actual (or phase) space. Therefore, the conclusions made in this article are universal and apply to stochastic objects and subjects of any scale and quality.

Let's consider such a chaotically wandering particle (ChWP) as a stochastic system distributed in a 3-dimensional space, with a conventionally distinguished center at the origin of the coordinate system (see Figure 1).

Suppose that the total mechanical energy of the ChWP $W(v, x, y, z, t)$ at a point with coordinates x, y, z and at time t is equal to

$$W(v, x, y, z, t) = T(v, x, y, z, t) + U(x, y, z, t) \pm \xi(x, y, z, t), \quad (1)$$

where $T(v, x, y, z, t)$ is the kinetic energy of a particle, which is determined by the velocity $v(v_x, v_y, v_z; x, y, z, t)$, at time t at a point with coordinates x, y, z (see Figure 1); $U(x, y, z, t)$ is the potential energy of a particle associated with the elasticity of its environment (or other factors) that, on average, tends to return this particle to the conditional center of the considered distributed stochastic system (see Figure 1); $\xi(x, y, z, t)$ is the energy that a particle receives or gives up for a short time when interacting with a chaotically fluctuating environment. This interaction energy can initially be spent on elastic deformations and/or a change in the angular velocity of rotation of a particle, but then it, in different fractions, randomly transforms into kinetic energy $T(v, x, y, z, t)$ and/or potential energy $U(x, y, z, t)$ of the ChWP.

The chaotically fluctuating environment surrounding the ChWP (i.e., the geometric center of a stochastic object or subject) can be a gas, a boiling liquid, a society, a securities market, etc. In the case of an elementary particle, a fluctuating space-time continuum (or a seething physical vacuum) can be considered a medium, as shown in the articles of E. Nelson [1-3].

Let's represent Ex. (1) in the form

$$W(v, x, y, z, t) \mp \xi(x, y, z, t) = T(v, x, y, z, t) + U(x, y, z, t). \quad (2)$$

Denote the left side of this expression as follows

$$E(x, y, z, t) = W(v, x, y, z, t) \mp \xi(x, y, z, t) \quad (3)$$

and we will call it the "mechanical energy" of the ChWP.

In this case, the all three energies of the ChWP: $E(x, y, z, t)$, $T(v, x, y, z, t)$ and $U(x, y, z, t)$ are random variables, but such that at each point of the considered region of space, the averaged equality is fulfilled

$$\langle E(x, y, z, t) \rangle = \langle T(v, x, y, z, t) \rangle + \langle U(x, y, z, t) \rangle, \quad (4)$$

$$\text{or } \langle T(v, x, y, z, t) \rangle + \langle U(x, y, z, t) \rangle - \langle E(x, y, z, t) \rangle = 0, \quad (5)$$

where $\langle \rangle$ – means local averaging, i.e., averaging in the vicinity of a point with coordinates x, y, z , located in the interior of a distributed stochastic system (see Figure 1).

Eq. (4) means that the additional energy $\pm \xi(x, y, z, t)$, which the particle receives or gives up when interacting with the bubbling environment, is randomly redistributed over time between its kinetic and potential energies.

If the speed of movement of the ChWP $v(v_x, v_y, v_z)$ is small compared to the speed of light, then according to nonrelativistic mechanics, it has kinetic energy

$$T(p_x, p_y, p_z, x, y, z, t) = T(v_x, v_y, v_z, x, y, z, t) = \frac{m[v_x^2(x, y, z, t) + v_y^2(x, y, z, t) + v_z^2(x, y, z, t)]}{2},$$

where $p_i = mv_i(x, y, z, t)$ are the instantaneous value of the i -th component of the ChWP momentum vector at time t and a point with coordinates x, y, z (Figure 1); m is the mass of the particle.

The form of the locally averaged potential energy of the ChWP $\langle U(x, y, z, t) \rangle$ is not specified at this stage of the study. As an example, one of the possible types of $\langle U(x, y, z, t) \rangle$ is used in §3.6 in [21](arXiv:1702.01880).

Let's integrate Eq. (5) over time

$$\int_{t_1}^{t_2} [\langle T(p_x, p_y, p_z, x, y, z, t) \rangle + \langle U(x, y, z, t) \rangle - \langle E(x, y, z, t) \rangle] dt = 0, \quad (6)$$

and we will call this expression «the locally averaged balance of the stochastic system».

To simplify mathematical calculations, let us consider a one-dimensional case without loss of generality (WLOG) of conclusions (in the case of three dimensions, only the number of integrations increases), while the locally averaged balance (6) takes on a simplified form

$$\langle S(x, t_1, t_2) \rangle = \int_{t_1}^{t_2} [\langle T(p_x, x, t) \rangle + \langle U(x, t) \rangle - \langle E(x, t) \rangle] dt, \quad (7)$$

where the random characteristics of the ChWP: $E(x, t)$, $T(x, t)$ and $U(x, t)$ meet the energy balance condition $\langle E(x, t) \rangle = \langle T(p_x, x, t) \rangle + \langle U(x, t) \rangle$, acting in the neighborhood of each point with coordinate x .

2.2 Globally Averaged Efficiency of the ChWP

Let's perform a global averaging of the locally averaged balance (7) over the entire region of the X -axis in which the ChWP moves (see Figure 1)

$$\overline{\langle S_x(t_1, t_2) \rangle} = \overline{\int_{t_1}^{t_2} [\langle T(p_x, x, t) \rangle + \langle U(x, t) \rangle - \langle E(x, t) \rangle] dt}. \quad (8)$$

Integration and averaging operations are commutative; therefore, Ex. (8) can be represented as

$$\overline{\langle S_x(t_1, t_2) \rangle} = \int_{t_1}^{t_2} [\overline{\langle T(p_x, x, t) \rangle} + \overline{\langle U(x, t) \rangle} - \overline{\langle E(x, t) \rangle}] dt. \tag{9}$$

We represent the globally averaged kinetic energy of the ChWP in the following form

$$\overline{\langle T(p_x, x, t) \rangle} = \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x, t) p_x^2 dp_x, \tag{10}$$

where $\rho(p_x, t)$ is the probability density function (PDF) of the x -component of the particle momentum p_x .

In Ex. (10), averaging occurs over all possible momenta p_x of the ChWP, regardless of the location of the particle. In general, $\rho(p_x, t)$ can vary with time t .

Let's represent the globally averaged potential energy $\overline{\langle U(x, t) \rangle}$ and the globally averaged mechanical energy $\overline{\langle E(x, t) \rangle}$ of the ChWP in the form

$$\overline{\langle U(x, t) \rangle} = \int_{-\infty}^{\infty} \rho(x, t) \langle U(x, t) \rangle dx, \tag{11}$$

$$\overline{\langle E(x, t) \rangle} = \int_{-\infty}^{\infty} \rho(x, t) \langle E(x, t) \rangle dx, \tag{12}$$

where $\rho(x, t)$ is the PDF of the projection of the ChWP position on the X axis (see Figure 1). In general, $\rho(x, t)$ can change with time t .

Substituting globally averaged quantities (10), (11), and (12) into Ex. (9), we obtain

$$\overline{\langle S_x(t_1, t_2) \rangle} = \int_{t_1}^{t_2} \left\{ \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x, t) p_x^2 dp_x + \int_{-\infty}^{\infty} \rho(x, t) [\langle U(x, t) \rangle - \langle E(x, t) \rangle] dx \right\} dt. \tag{13}$$

Let's exclude the unnecessary concept of "mass" from the proposed mathematical model. To do this, we introduce the mass-independent quantities:

$$\overline{\langle s_x(t_1, t_2) \rangle} = \frac{\overline{\langle S_x(t_1, t_2) \rangle}}{m} \tag{14}$$

is the globally averaged "x-efficiency" of the ChWP;

$$\langle \varepsilon(x, t) \rangle = \frac{\langle E(x, t) \rangle}{m} \tag{15}$$

is the locally averaged "x-mechanical energy" of the ChWP;

$$\langle u(x, t) \rangle = \frac{\langle U(x, t) \rangle}{m} \tag{16}$$

is the locally averaged "x-potential energy" of the ChWP;

$$\overline{\langle k(v_x, x, t) \rangle} = \frac{\overline{\langle T(p_x, x, t) \rangle}}{m} = \frac{1}{2m^2} \int_{-\infty}^{\infty} \rho(p_x, t) p_x^2 dp_x = \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x, t) v_x^2 dv_x \quad (17)$$

is the globally averaged “x-kinetic energy” of the ChWP.

Substituting Ex.s (14)-(17) into Ex. (13), we obtain the globally averaged x-efficiency of the ChWP, independent of the particle “mass”¹

$$\overline{\langle s_x(t_1, t_2) \rangle} = \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x, t) v_x^2 dv_x + \int_{-\infty}^{\infty} \rho(x, t) [\langle \varepsilon(x, t) \rangle - \langle u(x, t) \rangle] dx \right\} dt. \quad (18)$$

Note that if in the system under consideration there is no chaotic behavior of the particle, i.e. it behaves deterministically, then integral (9) takes the form

$$S_x(t_1, t_2) = \int_{t_1}^{t_2} T(p_x, x, t) + U(x, t) dt - E(x, t) \Big|_{t_1}^{t_2}, \quad (18a)$$

and we, in fact, return to the Lagrangian formalism of classical mechanics.

2.3 Stationary State of the ChWP

Consider a stationary globally averaged state of the stochastic system under study, i.e., when the averaged behavior of the ChWP does not depend on the time t .

In this case, the globally averaged x-efficiency of the ChWP (18) takes on a simplified form

$$\overline{\langle s_x(t_1, t_2) \rangle} = \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x) v_x^2 dv_x + \int_{-\infty}^{\infty} \rho(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle] dx \right\} dt. \quad (19)$$

Let’s represent Ex. (19) in coordinate form. To do this, we will perform the following actions:

1. Let’s write the PDF $\rho(x)$ in the form of the product of the two probability amplitude $\psi(x)$:

$$\rho(x) = \psi(x)\psi(x) = \psi^2(x). \quad (20)$$

2. Let’s use the coordinate representation of the averaged x-component of the ChWP velocity raised to the n -th power {see (A2.12) in Appendix 2}. In particular, for $n = 2$, we have

$$\overline{v_x^2} = \int_{-\infty}^{+\infty} \rho(v_x) v_x^2 dv_x = \int_{-\infty}^{+\infty} \psi(x) \left(\mp i \eta_x \frac{\partial}{\partial x} \right)^2 \psi(x) dx = -\eta_x^2 \int_{-\infty}^{\infty} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} dx, \quad (21)$$

where, according to (A2.13a) in Appendix 2

$$\eta_x = \frac{2\sigma_x^2}{\tau_{xcor}} = \text{constant} \quad (22)$$

¹ According to the author, the outdated heuristic concept of “mass” hinders the development of fully geometrized physics. Therefore, this concept should be gradually excluded from all physical representations of the surrounding reality. This article attempts to show how this can be done.

is a constant scale parameter; σ_x is the standard deviation of a stationary random process $x(t)$ associated with the projection of the position of the ChWP on the X axis (see Figure A1.1 in Appendix 1): τ_{xcor} is the autocorrelation interval of a given random process $x(t)$. As seen from Figure 1 and Figure A1.1 in Appendix 1, the autocorrelation interval shows how smooth the investigated random process is, how fast the ChWP is moving and changing direction. The larger the autocorrelation interval, the lower the speed of the ChWP and the slower the change in the direction of its movement.

3. Using Ex. (21), we represent the globally averaged x -kinetic energy of the ChWP (17) in the form

$$\overline{\langle k(v_x, x) \rangle} = \frac{\overline{\langle T(p_x, x) \rangle}}{m} = \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x) v_x^2 dv_x = -\frac{\eta_x^2}{2} \int_{-\infty}^{\infty} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} dx. \quad (23)$$

4. Substituting Ex.s (20) and (23) into integral (19), we obtain the functional of the globally averaged x -efficiency of the ChWP in the coordinate representation

$$\overline{\langle s_x(t_1, t_2) \rangle} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left(-\frac{\eta_x^2}{2} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} + \psi^2(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle] \right) dx dt. \quad (24)$$

2.4 Derivation of the Stationary Stochastic Euler-Poisson Equation

Let's find the equation for the extremals of Functional (24), i.e., and we define the condition for obtaining the functions $\psi(x)$ for which this function takes extreme values.

Since there are no time-dependent functions in Functional (24), we will seek the condition for the extremality of the inner functional

$$w_x = \int_{-\infty}^{\infty} \left(-\frac{\eta_x^2}{2} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} + \psi^2(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle] \right) dx. \quad (25)$$

In the calculus of variations, it was shown [22, 23] that the extremal $f(x)$ of a general functional

$$I(f) = \int_{-\infty}^{\infty} L(x, f, f', f'', \dots, f^{(k)}) dx, \quad (26)$$

where $f' := \frac{\partial f}{\partial x}$, $f'' := \frac{\partial^2 f}{\partial x^2}$, \dots , $f^{(k)} := \frac{\partial^k f}{\partial x^k}$ is determined by the generalized Euler-Lagrange equation [22, 23]

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial f'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial L}{\partial f''} \right) - \dots + (-1)^k \frac{\partial^k}{\partial x^k} \left(\frac{\partial L}{\partial f^{(k)}} \right) = 0. \quad (27)$$

However, it should be noted that under other boundary conditions, the extremal of the functional (26) is determined by the Euler-Poisson-Elsgolts equation ([24], P.P. 326–329, Elsgolts)

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x} \left\{ \frac{\partial L}{\partial f'} \right\} + \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial f''} \right\} - \dots + (-1)^k \frac{\partial^k}{\partial x^k} \left\{ \frac{\partial L}{\partial f^{(k)}} \right\} = 0, \quad (27a)$$

where $\frac{\partial}{\partial x} \left\{ \frac{\partial L}{\partial f'} \right\} = \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial f'} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f'} \right) \frac{\partial f}{\partial x} + \frac{\partial}{\partial f'} \left(\frac{\partial L}{\partial f'} \right) \frac{\partial f'}{\partial x} + \dots + \frac{\partial}{\partial f^{(k)}} \left(\frac{\partial L}{\partial f'} \right) \frac{\partial f^{(k)}}{\partial x}$ is the first total partial derivative concerning x ; $\frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial f''} \right\} = \frac{\partial^2}{\partial x^2} \left(\frac{\partial L}{\partial f''} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f''} \right) \frac{\partial^2 f}{\partial x^2} + \frac{\partial}{\partial f'} \left(\frac{\partial L}{\partial f''} \right) \frac{\partial^2 f'}{\partial x^2} + \dots + \frac{\partial}{\partial f^{(k)}} \left(\frac{\partial L}{\partial f''} \right) \frac{\partial^2 f^{(k)}}{\partial x^2}$ is the second total partial derivative concerning x , and so on.

In the case of searching for the extremal $f = \psi(x)$ of functional (25), we have a Lagrangian

$$L = -\frac{\eta_x^2}{2} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} + \psi^2(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle], \tag{28}$$

depending only on ψ and ψ'' , therefore, the generalized Euler-Lagrange equation (27) is simplified

$$\frac{\partial L}{\partial \psi} + \frac{\partial^2}{\partial x^2} \left(\frac{\partial L}{\partial \psi''} \right) = 0. \tag{29}$$

Because, in this case, all other terms in Eq. (27) are equal to zero.

We write out the terms from Eq. (29)

$$\frac{\partial L}{\partial \psi} = -\frac{\eta_x^2}{2} \frac{\partial^2 \psi(x)}{\partial x^2} + 2\psi(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle], \quad \frac{\partial L}{\partial \psi''} = -\frac{\eta_x^2}{2} \psi(x). \tag{30}$$

Substituting Ex.s (30) into Eq. (29), we obtain a one-dimensional stationary stochastic Euler-Poisson equation that does not depend on the mass of the particle

$$-\frac{\eta_x^2}{2} \frac{\partial^2 \psi(x)}{\partial x^2} + \psi(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle] = 0. \tag{31}$$

If instead of the Euler-Lagrange equation (27), we use the Euler-Poisson-Elsgolts equation (27a), then instead of Eq. (29) we get the equation

$$\frac{\partial L}{\partial \psi} + \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial \psi''} \right\} = 0, \tag{31a}$$

where $\frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial \psi''} \right\} = \frac{\partial^2}{\partial x^2} \left(\frac{\partial L}{\partial \psi''} \right) + \frac{\partial}{\partial \psi} \left(\frac{\partial L}{\partial \psi''} \right) \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial}{\partial \psi''} \left(\frac{\partial L}{\partial \psi''} \right) \frac{\partial^2 \psi''}{\partial x^2}$.

Substituting Lagrangian (28) into Eq. (31a), in this case, we obtain the following equation for extremals $\psi(x)$ of Functional (25)

$$-\frac{3\eta_x^2}{4} \frac{\partial^2 \psi(x)}{\partial x^2} + \psi(x) [\langle u(x) \rangle - \langle \varepsilon(x) \rangle] = 0. \tag{31b}$$

Eqs. (31) and (31b) differ by a factor of 3/2 in front of the first term. Further research has shown that the Eq. (31b) is preferable (see § 2.9).

Consider now the 3-dimensional case. Let's represent Functional (25) in an expanded form

$$\overline{\langle S_r \rangle} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(-\frac{\eta_r^2}{2} \psi(x, y, z) \nabla^2 \psi(x, y, z) + \psi^2(x, y, z) [\langle u(x, y, z) \rangle - \langle \varepsilon(x, y, z) \rangle] \right) dx dy dz dt, \tag{32}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator;

$$\eta_r = \frac{2\sigma_r^2}{\tau_{rcor}} = constant \tag{33}$$

is a constant scale parameter, here for the three-dimensional case:

$$\sigma_r = \frac{1}{\sqrt{3}} \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \tag{34}$$

is the standard deviation of the random 3-dimensional trajectory of the ChWP from the conditional center of the considered stochastic system (see Figure 1);

$$\tau_{rcor} = \frac{1}{3} (\tau_{xcor} + \tau_{ycor} + \tau_{zcor}) \tag{35}$$

is the autocorrelation interval of a given 3-dimensional stationary random process; see the explanation for Ex. (22).

For the 3-dimensional case, the general functional (26) has the form

$$I[f] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x_1, x_2, x_3, f, f_1, f_2, f_3, f_{11}, f_{22}, f_{33}) dx_1 dx_2 dx_3, \tag{35a}$$

where

$$f_i := \frac{\partial f}{\partial x_i}, \quad f_{ii} := \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2} \quad (i=1,2,3). \tag{35b}$$

The extremal $f(x_1, x_2, x_3)$ of this functional is determined by the Euler-Poisson-Elsgolts equation ([24], see P.P. 325-329, The Calculus of Variations: L. Elsgolts)

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x_1} \left\{ \frac{\partial L}{\partial f_1} \right\} - \frac{\partial}{\partial x_2} \left\{ \frac{\partial L}{\partial f_2} \right\} - \frac{\partial}{\partial x_3} \left\{ \frac{\partial L}{\partial f_3} \right\} + \frac{\partial^2}{\partial x_1^2} \left\{ \frac{\partial L}{\partial f_{11}} \right\} + \frac{\partial^2}{\partial x_2^2} \left\{ \frac{\partial L}{\partial f_{22}} \right\} + \frac{\partial^2}{\partial x_3^2} \left\{ \frac{\partial L}{\partial f_{33}} \right\} = 0, \tag{35c}$$

where

$$\begin{aligned} \frac{\partial}{\partial x_i} \left\{ \frac{\partial L}{\partial f_i} \right\} &= \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial f_i} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f}{\partial x_i} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_1}{\partial x_i} + \frac{\partial}{\partial f_2} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_2}{\partial x_i} + \frac{\partial}{\partial f_3} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_3}{\partial x_i} \\ &+ \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{11}}{\partial x_i} + \frac{\partial}{\partial f_{22}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{22}}{\partial x_i} + \frac{\partial}{\partial f_{33}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{33}}{\partial x_i} \end{aligned} \tag{35d}$$

is the first total partial derivative concerning x_i (where $i = 1,2,3$);

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} \left\{ \frac{\partial L}{\partial f_{ii}} \right\} &= \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial L}{\partial f_{ii}} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_1}{\partial x_i^2} + \frac{\partial}{\partial f_2} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_2}{\partial x_i^2} + \frac{\partial}{\partial f_3} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_3}{\partial x_i^2} \\ &+ \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{11}}{\partial x_i^2} + \frac{\partial}{\partial f_{22}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{22}}{\partial x_i^2} + \frac{\partial}{\partial f_{33}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{33}}{\partial x_i^2} \end{aligned} \tag{35e}$$

is the second entire partial derivative concerning x_i (where $i = 1,2,3$).

From the Functional (32), we have the Lagrangian:

$$L = -\frac{\eta_r^2}{2} \psi(x, y, z) \left(\frac{\partial^2 \psi(x, y, z)}{\partial x^2} + \frac{\partial^2 \psi(x, y, z)}{\partial y^2} + \frac{\partial^2 \psi(x, y, z)}{\partial z^2} \right) + \psi^2(x, y, z) [\langle u(x, y, z) \rangle - \langle \varepsilon(x, y, z) \rangle], \quad (35f)$$

where in $x_1 = x$, $x_2 = y$, $x_3 = z$; $f(x_1, x_2, x_3) = \psi(x, y, z)$,

$$\psi_x := \frac{\partial \psi}{\partial x}, \quad \psi_y := \frac{\partial \psi}{\partial y}, \quad \psi_z := \frac{\partial \psi}{\partial z}, \quad \psi_{xx} := \frac{\partial^2 \psi}{\partial x^2}, \quad \psi_{yy} := \frac{\partial^2 \psi}{\partial y^2}, \quad \psi_{zz} := \frac{\partial^2 \psi}{\partial z^2}. \quad (35g)$$

Since Lagrangian (35f) depends only on $\psi, \psi_{xx}, \psi_{yy}, \psi_{zz}$, the Euler-Poisson-Elsgolts equation (35c) is simplified

$$\frac{\partial L}{\partial \psi} + \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial \psi_{xx}} \right\} + \frac{\partial^2}{\partial y^2} \left\{ \frac{\partial L}{\partial \psi_{yy}} \right\} + \frac{\partial^2}{\partial z^2} \left\{ \frac{\partial L}{\partial \psi_{zz}} \right\} = 0, \quad (35h)$$

because all other terms in Eq. (35c) are equal to zero.

Taking into account expression (35e), we write the terms from equation (35h)

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} \left\{ \frac{\partial L}{\partial \psi_{ii}} \right\} &= \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial L}{\partial \psi_{ii}} \right) + \frac{\partial}{\partial \psi} \left(\frac{\partial L}{\partial \psi_{ii}} \right) \frac{\partial^2 \psi}{\partial x_i^2} + \frac{\partial}{\partial \psi_{xx}} \left(\frac{\partial L}{\partial \psi_{ii}} \right) \frac{\partial^2 \psi_{xx}}{\partial x_i^2} + \\ &+ \frac{\partial}{\partial \psi_{yy}} \left(\frac{\partial L}{\partial \psi_{ii}} \right) \frac{\partial^2 \psi_{yy}}{\partial x_i^2} + \frac{\partial}{\partial \psi_{zz}} \left(\frac{\partial L}{\partial \psi_{ii}} \right) \frac{\partial^2 \psi_{zz}}{\partial x_i^2}, \quad (\text{where } i = 1,2,3) \end{aligned} \quad (35h, a)$$

which are simplified to the expression

$$\frac{\partial^2}{\partial x_i^2} \left\{ \frac{\partial L}{\partial \psi_{ii}} \right\} = \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial L}{\partial \psi_{ii}} \right) + \frac{\partial}{\partial \psi} \left(\frac{\partial L}{\partial \psi_{ii}} \right) \frac{\partial^2 \psi}{\partial x_i^2}, \quad (\text{where } i = 1,2,3) \quad (35h, b)$$

since the rest of the terms in (35h,a) are equal to zero.

Let's substitute the Lagrangian (35f) into equation (35h) and preliminarily write out the terms from this equation, taking into account the expression (35h,b),

$$\begin{aligned} \frac{\partial L}{\partial \psi} &= -\frac{\eta_r^2}{2} \left(\frac{\partial^2 \psi(x, y, z)}{\partial x^2} + \frac{\partial^2 \psi(x, y, z)}{\partial y^2} + \frac{\partial^2 \psi(x, y, z)}{\partial z^2} \right) \\ &+ 2\psi(x, y, z) [\langle u(x, y, z) \rangle - \langle \varepsilon(x, y, z) \rangle], \end{aligned} \quad (35i)$$

$$\frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial \psi_{xx}} \right\} = -\frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial x^2} + \left(-\frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial x^2} \right) = -2 \frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial x^2},$$

$$\frac{\partial^2}{\partial y^2} \left\{ \frac{\partial L}{\partial \psi_{yy}} \right\} = -\frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial y^2} + \left(-\frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial y^2} \right) = -2 \frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial y^2},$$

$$\frac{\partial^2}{\partial z^2} \left\{ \frac{\partial L}{\partial \psi_{zz}} \right\} = -\frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial z^2} + \left(-\frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial z^2} \right) = -2 \frac{\eta_r^2}{2} \frac{\partial^2 \psi(x, y, z)}{\partial z^2}.$$

As a result of the substitution of expressions (35i) into the Euler-Poisson-Elsgolts equation (35h), we obtain the following three-dimensional stochastic equation

$$-\frac{3\eta_r^2}{2} \left\{ \frac{\partial^2 \psi(x, y, z)}{\partial x^2} + \frac{\partial^2 \psi(x, y, z)}{\partial y^2} + \frac{\partial^2 \psi(x, y, z)}{\partial z^2} \right\} + 2[\langle u(x, y, z) \rangle - \langle \varepsilon(x, y, z) \rangle] \psi(x, y, z) = 0. \quad (36)$$

Let's multiply both sides of this by -1 and write it in a compact form

$$\frac{3\eta_r^2}{4} \nabla^2 \psi(\vec{r}) + [\langle \varepsilon(\vec{r}) \rangle - \langle u(\vec{r}) \rangle] \psi(\vec{r}) = 0, \quad (37)$$

where $\psi(\vec{r}) = \psi(x, y, z)$ is the extremal of the Functional of the 3-dimensional globally averaged efficiency of the ChWP (32).

\vec{r} is the radius vector, with the beginning at the conditional center of the investigated stochastic system (see Figure 1) and the end at the point with coordinates x, y, z ; the modulus of this vector is

$$|\vec{r}| = (x^2 + y^2 + z^2)^{1/2}. \quad (38)$$

Eq. (37) will be called "the mass-independent stationary three-dimensional stochastic Euler-Poisson equation" (or abbreviated as "the stationary stochastic Euler-Poisson equation").

2.5 Stationary Stochastic Schrödinger-Euler-Poisson Equation

Consider the case when the locally averaged mechanical energy of the ChWP $\langle \varepsilon(\vec{r}) \rangle$ is constant at all points of the investigated stochastic system

$$\langle \varepsilon(\vec{r}) \rangle = \varepsilon = \text{constant}. \quad (39)$$

In such a stationary stochastic system, the locally averaged kinetic energy of the ChWP $\langle k(x, y, z, t) \rangle$ and its locally averaged potential energy $\langle u(x, y, z, t) \rangle$ so, on average, they pass into each other that their averaged sum at each point $\vec{r} := (x, y, z)$ always remains constant

$$\langle k(\vec{r}) \rangle + \langle u(\vec{r}) \rangle = \varepsilon = \text{constant}. \quad (40)$$

In this case, Eq. (37) takes the form

$$-\frac{3\eta_r^2}{4} \nabla^2 \psi(\vec{r}) + \langle u(\vec{r}) \rangle \psi(\vec{r}) = \varepsilon \psi(\vec{r}). \quad (41)$$

We compare stochastic Eq. (41) with the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + U(\vec{r}) \psi(\vec{r}) = E \psi(\vec{r}), \quad (42)$$

where \hbar is the reduced Planck's constant ($\hbar = 1.055 \times 10^{-34}$ J/Hz).

First, we divide both sides of Schrödinger equation (42) by the particle mass m

$$\frac{\hbar^2}{2m^2} \nabla^2 \psi(\vec{r}) + \frac{U(\vec{r})}{m} \psi(\vec{r}) = \frac{E}{m} \psi(\vec{r}). \quad (43)$$

We write Eq. (43) in the following form

$$-\frac{1}{2}\left(\frac{\hbar}{m}\right)^2 \nabla^2\psi(\vec{r})+u(\vec{r})\psi(\vec{r}) = \varepsilon\psi(\vec{r}), \tag{44}$$

where, according to the terminology introduced above (14)–(16):

$$\varepsilon = E/m \quad \text{is the “mechanical energality” of the particle;} \tag{45}$$

$$u(\vec{r}) = U(\vec{r})/m \quad \text{is the “potential energality” of the particle.} \tag{46}$$

Comparing Eqs. (41) and (44), we find that for

$$\frac{\hbar}{m} = \sqrt{\frac{3}{2}}\eta_r = \sqrt{\frac{3}{2}}\frac{2\sigma_r^2}{\tau_{rcor}} \quad \text{and} \quad u(\vec{r}) = \langle u(\vec{r}) \rangle \tag{47}$$

these equations coincide entirely.

The ratio of the volume of the cylinder to the volume of the ball inscribed in it (see Figure 2) is equal to 3/2. Archimedes was so shocked by this fact that he requested his relatives to engrave a sphere inscribed in a cylinder on his tombstone. It is believed that Cicero found Archimedes' grave thanks to this symbol (note by S. Petukhov).

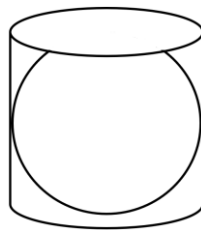


Figure 2 Sphere inscribed in a cylinder.

Thus, it turned out that the time-independent Schrödinger equation (42) is a particular case of the stochastic equation (41).

Therefore, Eq. (41) will be called “the mass-independent stationary three-dimensional stochastic Schrödinger-Euler-Poisson equation” (or abbreviated, “stationary stochastic Schrödinger-Euler-Poisson equation”).

Due to the smallness of the reduced Planck constant \hbar , the requirement $\eta_r = \hbar/m$ refers to stochastic systems at the microscopic level or extremely precise macro-level systems, such as the LIGO project.

At the same time, in all the previous reasoning, no restrictions were imposed on the scale parameter $\eta_r = 2\sigma_r^2/\tau_{rcor}$ (see Appendix 1 and Appendix 2). This parameter depends on the standard deviation σ_r and the autocorrelation coefficient τ_{rcor} of a random process, in which ChWP of any scale is involved. Therefore, stationary stochastic Eq.s (37) and (41) are suitable for describing the averaged states of stationary stochastic systems regardless of their size and quality.

Therefore, the stationary (i.e., time-independent) stochastic Eq.s (37) and (41) are suitable for describing the averaged states of stationary stochastic systems, regardless of their size and type of

association with their environment. In other words, stochastic Eq.s (37) and (41) are applicable to describe the averaged states of compact objects or subjects wandering chaotically in 3-dimensional space (actual or phase) in physics, biology, sociology, psychology, economics, etc.

Section 3.6 in [21](arXiv:1702.01880) gives an example of using the stationary stochastic Eq. (41) to describe a discrete (quantum) set of averaged states of a chaotically oscillating nucleus of a biological cell.

Eq. (41) is similarly applicable to describe stationary quantum states of many other macroscopic stochastic systems, such as, for example, a chaotically shifting geometric center of the planet's core and many other aforementioned randomly moving centers of various objects.

In contrast to Eq. (42), which was hypothesized by Erwin Schrödinger in 1926, stochastic Eq.s (37) and (41) are obtained based on the variational principle. That is, stochastic Eq.s (37) and (41) are the conditions for searching for extremals of the globally averaged “efficiency” functional of ChWP (32), which is symmetric concerning the simultaneous tendency of any stochastic system to “order” and “chaos”. This is the additional significance of the result obtained.

Let’s consider, for example, the possibility of using the obtained formulas to study white noise.

White noise is a mathematical abstraction. That is, this is an ideal stationary random process with an infinite spectrum

$$S(\omega) = \sigma_{\omega}^2,$$

where σ_{ω}^2 is white noise dispersion, and correlation function

$$R_{acor}(\tau) = \sigma_{\omega}^2 \delta(\tau),$$

where $\tau = t_2 - t_1$, $\delta(\tau)$ is delta function.

This means that the autocorrelation coefficient of white noise tends to be zero.

$$\tau_{\omega acor} \rightarrow 0.$$

Therefore, white noise cannot be studied within the scope of this article. However, if we assume that the stationary random process differs negligibly from white noise, i.e., if its autocorrelation coefficient τ_{acor} is negligible but not equal to zero

$$0 \cong \tau_{sacor} \neq 0.$$

Then, such a slightly “darkened” random process can be described by the stochastic equation (41) with $\varepsilon = \sigma_{\omega}^2 / \tau_{sacor}^2$ and $\langle u(\vec{r}) \rangle = 0$

$$\frac{3\eta_r^2}{4} \nabla^2 \psi(\vec{r}) + \frac{\sigma_{\omega}^2}{\tau_{sacor}^2} \psi(\vec{r}) = 0, \tag{47a}$$

where according to expression (47) $\eta_r = \frac{2\sigma_{\omega}^2}{\tau_{sacor}} = \sqrt{\frac{2}{3}} \frac{\hbar}{m}$.

This equation can be represented as follows

$$\nabla^2 \psi(\vec{r}) + \frac{1}{3\sigma_{\omega}^2} \psi(\vec{r}) = 0. \tag{47b}$$

Let's consider one section of the 3-dimensional random process under study; as a result, we obtain a one-dimensional stationary random process. For the one-dimensional case, Eq. (47b) takes the form

$$\nabla^2\psi(x) + \frac{1}{\sigma_\omega^2}\psi(x) = 0. \tag{47c}$$

This is a particular case of the well-known quantum harmonic oscillator equation. The solution to this equation is known

$$\psi(x) = \frac{1}{\sqrt{\sigma_\omega^2\sqrt{\pi}}}\exp\left(-\frac{x^2}{2\sigma_\omega^2}\right).$$

Thus, it turned out that in any section, the "darkened" white noise under study is a Gaussian stationary random process with a distribution density

$$\rho(x) = \psi(x)\psi(x) = \frac{1}{\sqrt{\pi\sigma_\omega^2}}\exp\left(-\frac{x^2}{\sigma_\omega^2}\right). \tag{47d}$$

This result confirms that the resulting stochastic equation is suitable for studying various stationary random processes.

2.6 Time-Dependent Stochastic Euler-Poisson Equation

Let the averaged characteristics of the random trajectory of the ChWP movement change with time, but so slowly that in each small time interval Δt , all these characteristics can be considered unchanged. Such an unstable behavior of a particle is considered in Appendix 1 and is called a pseudo-stationary random process (PSRP).

Let's also assume that for such a pseudo-stationary stochastic system (i.e., the non-equilibrium state of the ChWP), its globally averaged mechanical energality (GAME) changes insignificantly over time

$$\overline{\langle \varepsilon(x, y, z, t) \rangle} = \overline{\langle k(x, y, z, t) \rangle} + \overline{\langle u(x, y, z, t) \rangle}. \tag{48}$$

Since the change in GAME is slow, we can write

$$\overline{\langle \varepsilon(x, y, z, t) \rangle} = \overline{\langle \varepsilon(x, y, z, t_0) \rangle} \pm \overline{\langle \varepsilon_k(x, y, z, t) \rangle}, \tag{49}$$

where $\overline{\langle \varepsilon(x, y, z, t_0) \rangle}$ is the initial value of the globally averaged mechanical energality of the ChWP at the time t_0 ; $\pm \overline{\langle \varepsilon_k(x, y, z, t) \rangle}$ is a slight change {increase (+) or decrease (-)} of the globally averaged mechanical energality of the ChWP for a time interval t , associated with a change in its kinetic energality Δk .

To reduce the calculations, consider the one-dimensional case without loss of generality of conclusions for the case of 3-dimensions, and represent Ex. (49) in an abbreviated form

$$\overline{\langle \varepsilon(x, t) \rangle} = \overline{\langle \varepsilon(x, t_0) \rangle} \pm \overline{\langle \varepsilon_k(x, t) \rangle}. \tag{50}$$

Let's write Ex. (50) as follows

$$\overline{\langle \varepsilon(x, t) \rangle} = \int_{-\infty}^{\infty} \rho(x, t) \langle \varepsilon(x, t_0) \rangle dx \pm \int_{-\infty}^{\infty} \rho(\varepsilon_{kx}, t) \varepsilon_{kx} d\varepsilon_{kx}, \quad (51)$$

where ε_{kx} is a local change in the mechanical energy of the ChWP due to a slight change in its kinetic energy in the direction of the X -axis (see § A2.2 in Appendix 2); $\rho(\varepsilon_{kx}, t)$ is the probability distribution function (PDF) of changes in the mechanical energy of the ChWP ε_{kx} .

Let's substitute Ex. (51) into the globally averaged x -efficiency of the ChWP (18)

$$\overline{\langle s_x(\Delta t) \rangle} = \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x, t) v_x^2 dv_x + \int_{-\infty}^{\infty} \rho(x, t) [\langle u(x, t) \rangle - \langle \varepsilon(x, t_0) \rangle] dx \pm \int_{-\infty}^{\infty} \rho(\varepsilon_{kx}, t) \varepsilon_{kx} d\varepsilon_{kx} \right\} dt. \quad (52)$$

Let's write Ex. (52) in the coordinate representation. For this, we express the PDFs $\rho(x, t)$, $\rho(v_x, t)$ and $\rho(\varepsilon_x, t)$ in terms of the probability amplitude $\psi(x, t)$.

According to Ex. (A1.46) in Appendix 1

$$\rho(x, t) = \psi(x, t)\psi(x, t) = \psi^2(x, t), \quad (53)$$

According to Ex. (A2.13) in Appendix 2

$$\frac{1}{2} \overline{v_x^2(t)} = \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x, t) v_x^2 dv_x = -\frac{\eta_x^2}{2} \int_{-\infty}^{\infty} \psi(x, t) \frac{\partial^2 \psi(x, t)}{\partial x^2} dx, \quad (54)$$

where {see Ex. (A2.13a) in Appendix 2}

$$\eta_x = \frac{2\sigma_x^2(t - t_0)}{\tau_{x\text{cor}}(t - t_0)} \approx \frac{2\sigma_x^2}{\tau_{x\text{cor}}} = \text{constant}.$$

is a constant scale parameter², with dimensions (m²/s).

According to Ex. (A2.31) in Appendix 2

$$\overline{\langle \varepsilon_k(x, t) \rangle} = \int_{-\infty}^{\infty} \rho(\varepsilon_{kx}, t) \varepsilon_{kx} d\varepsilon_{kx} = \pm i \frac{\eta_x^2}{D} \int_{-\infty}^{+\infty} \psi(x, t) \frac{\partial \psi(x, t)}{\partial t} dx, \quad (55)$$

where D is the imaginary part of the complex self-diffusion coefficient $B = iD$ of a chaotically wandering particle (ChWP) with dimension m²/s, {see Ex. (A2.19) in Appendix 2}. It is assumed that $D = \text{constant}$.

As well as

² The invariability of the scale parameter $\eta_x = \frac{2\sigma_x^2(t)}{\tau_{x\text{cor}}(t)} \approx \text{constant}$ with a change in the wave function $\psi(x, t)$ in this case is due to the fact that the variance $\sigma_x^2(t)$ and the autocorrelation interval $\tau_{x\text{cor}}(t)$ of the stochastic system under study change (i.e., increase or decrease) practically simultaneously and in the same way. Keeping unchanged the ratio of the main averaged characteristics of the investigated random process $\sigma_x^2(t)/\tau_{x\text{cor}}(t)$ in Appendix 2 is called: "The law of proportional constancy of the scale parameter η_x of the stochastic system." This effect is similar to the phenomenon described in the molecular kinetic theory, when the expansion of a gas is invariably accompanied by a slowdown in the speed of motion of its atoms and/or molecules and a decrease in the intensity of their collisions (i.e., a change in the direction of their motion); and, conversely, gas compression is accompanied by an increase in the speed and intensity of collisions of its atoms and/or molecules.

$$\int_{-\infty}^{\infty} \rho(x, t) \langle u(x, t) \rangle dx = \int_{-\infty}^{\infty} \psi^2(x, t) \langle u(x, t) \rangle dx, \tag{56}$$

$$\int_{-\infty}^{\infty} \rho(x, t) \langle \varepsilon(x, t_0) \rangle dx = \int_{-\infty}^{\infty} \psi^2(x, t) \langle \varepsilon(x, t_0) \rangle dx. \tag{57}$$

Substituting Ex.s (53)-(57) into integral (52), we obtain the coordinate representation of the pseudo-stationary globally averaged x -efficiency of the ChWP

$$\overline{\langle s_x(t) \rangle} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left(-\frac{\eta_x^2}{2} \psi(x, t) \frac{\partial^2 \psi(x, t)}{\partial x^2} + \psi^2(x, t) [\langle u(x, t) \rangle - \langle \varepsilon(x, t_0) \rangle] \pm \frac{\eta_x^2}{D} \psi(x, t) \frac{\partial \psi(x, t)}{\partial t} \right) dx dt. \tag{58}$$

In the case of 3-dimensional consideration, this function has the form

$$\overline{\langle s_r(t) \rangle} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(-\frac{\eta_r^2}{2} \psi(\vec{r}, t) \nabla^2 \psi(\vec{r}, t) + [\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t_0) \rangle] \psi^2(\vec{r}, t) \pm i \frac{\eta_r^2}{D} \psi(\vec{r}, t) \frac{\partial \psi(\vec{r}, t)}{\partial t} \right) dx dy dz dt, \tag{59}$$

where $\vec{r} := (x, y, z)$.

Let's find the equation for the extremals $\psi(x, y, z, t)$ of Functional (59).

First, recall that the extremality condition for a functional of the form

$$I = \int_{x_{01}}^{x_{02}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x_0, x_1, x_2, x_3, f, f_0, f_1, f_2, f_3, f_{00}, f_{11}, f_{22}, f_{33}) dx_0 dx_1 dx_2 dx_3$$

where $f_i := \frac{\partial f}{\partial x_i}, f_{ii} := \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2} \quad (i = 0, 1, 2, 3)$ (60)

is determined by the Euler-Poisson-Elsgolts equation [24](see P.P. 326, The Calculus Of Variations: L. Elsgolts)

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x_0} \left\{ \frac{\partial L}{\partial f_0} \right\} - \frac{\partial}{\partial x_1} \left\{ \frac{\partial L}{\partial f_1} \right\} - \frac{\partial}{\partial x_2} \left\{ \frac{\partial L}{\partial f_2} \right\} - \frac{\partial}{\partial x_3} \left\{ \frac{\partial L}{\partial f_3} \right\} + \frac{\partial^2}{\partial x_0^2} \left\{ \frac{\partial L}{\partial f_{00}} \right\} + \frac{\partial^2}{\partial x_1^2} \left\{ \frac{\partial L}{\partial f_{11}} \right\} + \frac{\partial^2}{\partial x_2^2} \left\{ \frac{\partial L}{\partial f_{22}} \right\} + \frac{\partial^2}{\partial x_3^2} \left\{ \frac{\partial L}{\partial f_{33}} \right\} = 0, \tag{61}$$

where

$$\frac{\partial}{\partial x_i} \left\{ \frac{\partial L}{\partial f_i} \right\} = \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial f_i} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f}{\partial x_i} + \frac{\partial}{\partial f_0} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_0}{\partial x_i} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_1}{\partial x_i} + \frac{\partial}{\partial f_2} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_2}{\partial x_i} + \frac{\partial}{\partial f_3} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_3}{\partial x_i} + \frac{\partial}{\partial f_{00}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{00}}{\partial x_i} + \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{11}}{\partial x_i} + \frac{\partial}{\partial f_{22}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{22}}{\partial x_i} + \frac{\partial}{\partial f_{33}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{33}}{\partial x_i} \tag{62}$$

is the first complete partial derivative concerning x_i ($i = 0, 1, 2, 3$);

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} \left\{ \frac{\partial L}{\partial f_{ii}} \right\} &= \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial L}{\partial f_{ii}} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial}{\partial f_0} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_0}{\partial x_i^2} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_1}{\partial x_i^2} \\ &+ \frac{\partial}{\partial f_2} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_2}{\partial x_i^2} + \frac{\partial}{\partial f_3} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_3}{\partial x_i^2} + \frac{\partial}{\partial f_{00}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{00}}{\partial x_i^2} + \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{11}}{\partial x_i^2} \\ &+ \frac{\partial}{\partial f_{22}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{22}}{\partial x_i^2} + \frac{\partial}{\partial f_{33}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{33}}{\partial x_i^2} \end{aligned} \tag{63}$$

is the second complete partial derivative concerning x_i ($i = 0,1,2,3$);

In the case of functional (59), we have Lagrangian:

$$\begin{aligned} L &= -\frac{\eta_r^2}{2} \psi(x, y, z, t) \left(\frac{\partial^2 \psi(x, y, z, t)}{\partial x^2} + \frac{\partial^2 \psi(x, y, z, t)}{\partial y^2} + \frac{\partial^2 \psi(x, y, z, t)}{\partial z^2} \right) \\ &+ [\langle u(x, y, z, t) \rangle - \langle \varepsilon(x, y, z, t) \rangle] \psi^2(x, y, z, t) \pm i \frac{\eta_r^2}{D} \psi(x, y, z, t) \frac{\partial \psi(x, y, z, t)}{\partial t}, \end{aligned} \tag{64}$$

where in

$$x_1 = t, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z; \quad f(x_0, x_1, x_2, x_3) = \psi(x, y, z, t), \tag{65}$$

$$\psi_t := \frac{\partial \psi}{\partial t}, \quad \psi_x := \frac{\partial \psi}{\partial x}, \quad \psi_y := \frac{\partial \psi}{\partial y}, \quad \psi_z := \frac{\partial \psi}{\partial z},$$

$$\psi_{tt} := \frac{\partial^2 \psi}{\partial t^2}, \quad \psi_{xx} := \frac{\partial^2 \psi}{\partial x^2}, \quad \psi_{yy} := \frac{\partial^2 \psi}{\partial y^2}, \quad \psi_{zz} := \frac{\partial^2 \psi}{\partial z^2}.$$

Since Lagrangian (64) depends only on $\psi, \psi_t, \psi_{xx}, \psi_{yy}$ and ψ_{zz} , the Euler-Poisson-Elsgolts equation (61) is simplified

$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial t} \left\{ \frac{\partial L}{\partial \psi_t} \right\} + \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial \psi_{xx}} \right\} + \frac{\partial^2}{\partial y^2} \left\{ \frac{\partial L}{\partial \psi_{yy}} \right\} + \frac{\partial^2}{\partial z^2} \left\{ \frac{\partial L}{\partial \psi_{zz}} \right\} = 0. \tag{66}$$

Because all other terms in (61) are equal to zero.

As a result of substituting the Lagrangian (64) into expressions (62) and (63), taking into account the notation (65), we obtain the following terms in equation (66)

$$\frac{\partial L}{\partial \psi} = -\frac{\eta_r^2}{2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial x^2} + 2\psi(\vec{r}, t) [\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t_0) \rangle] \pm i \frac{\eta_r^2}{D} \frac{\partial \psi(\vec{r}, t)}{\partial t},$$

$$\frac{\partial}{\partial t} \left\{ \frac{\partial L}{\partial \psi_t} \right\} = \pm i 2 \frac{\eta_r^2}{D} \frac{\partial \psi(\vec{r}, t)}{\partial t}, \quad \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial L}{\partial \psi_{xx}} \right\} = -2 \frac{\eta_r^2}{2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial x^2},$$

$$\frac{\partial^2}{\partial y^2} \left\{ \frac{\partial L}{\partial \psi_{yy}} \right\} = -2 \frac{\eta_r^2}{2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial x^2}, \quad \frac{\partial^2}{\partial z^2} \left\{ \frac{\partial L}{\partial \psi_{zz}} \right\} = -2 \frac{\eta_r^2}{2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial x^2}.$$

Substituting these expressions into the Euler-Poisson-Elsgolts (66), we obtain the desired mass-independent three-dimensional stochastic equation for determining the extremals $\psi(\vec{r}, t) = \psi(x, y, z, t)$ of the functional of the globally averaged ChWP efficiency (59)

$$\pm i \frac{\eta_r^2}{D} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{3\eta_r^2}{2} \nabla^2 \psi(\vec{r}, t) + 2[\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t_0) \rangle] \psi(\vec{r}, t). \quad (67)$$

Eq. (67) is a system of two equations

$$\begin{cases} -i \frac{\eta_r^2}{2D} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{3\eta_r^2}{4} \nabla^2 \psi(\vec{r}, t) + [\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t_0) \rangle] \psi(\vec{r}, t), & (67a) \\ i \frac{\eta_r^2}{2D} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{3\eta_r^2}{4} \nabla^2 \psi(\vec{r}, t) + [\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t_0) \rangle] \psi(\vec{r}, t). & (67b) \end{cases}$$

This suggests that in a slowly varying (i.e., pseudo-stationary) stochastic system, two processes should co-occur, as described by the two Eqs. (67a) and (67b).

Eq. (67) will be called “the mass-independent pseudo-stationary (i.e., time-dependent) three-dimensional stochastic Euler-Poisson equation” (or, in short, “the time-dependent stochastic Euler-Poisson equation”).

2.7 Time-Dependent Stochastic Schrödinger-Euler-Poisson Equation

Let's suppose that at the initial moment of time t_0 the globally averaged mechanical energy of the ChWP is zero (i.e., $\langle \varepsilon(\vec{r}, t_0) \rangle = 0$), then equation (67) takes the form

$$\pm i \frac{\eta_r^2}{2D} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{3}{4} \eta_r^2 \nabla^2 \psi(\vec{r}, t) + \langle u(\vec{r}, t) \rangle \psi(\vec{r}, t). \quad (68)$$

In the case when $D = \eta_r$ we obtain the self-consistent stochastic Euler-Poisson equation

$$\pm i \frac{\eta_r}{2} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{3}{4} \eta_r^2 \nabla^2 \psi(\vec{r}, t) + \langle u(\vec{r}, t) \rangle \psi(\vec{r}, t). \quad (68a)$$

Let's compare Eq. (68a) with the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + U(\vec{r}, t) \psi(\vec{r}, t). \quad (69)$$

First, we divide both sides of Eq. (69) by the particle mass m ; as a result, taking into account definition (46), we obtain

$$i \frac{\hbar}{m} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{1}{2} \frac{\hbar^2}{m^2} \nabla^2 \psi(\vec{r}, t) + u(\vec{r}, t) \psi(\vec{r}, t), \quad (70)$$

where $u(\vec{r}, t) = U(\vec{r}, t)/m$ is the potential energy of the ChWP.

Obviously, for $\langle u(\vec{r}) \rangle = u(\vec{r})$ and $\frac{\hbar}{m} = \eta_r = \frac{2\sigma_r^2}{\tau_{rcor}}$ self-consistent stochastic Eq. (68a) and Schrödinger equation (70) coincide with constant coefficients and the \pm sign.

It is interesting to note that Erwin Schrödinger wrote Eq. (4'') in "Quantisierung als Eigenwertproblem, Vierte Mitteilung", *Annalen der Physik* (1926) [25] in the following form

$$\Delta\psi - \frac{8\pi^2}{h^2}V\psi \pm \frac{4\pi i}{h} \frac{\partial\psi}{\partial t} = 0. \quad (71)$$

Let's rearrange the terms in this expression and take into account that $\hbar = h/2\pi$,

$$\pm i\hbar \frac{\partial\psi}{\partial t} = -\frac{1}{2}\hbar^2\nabla^2\psi + V\psi. \quad (72)$$

In this case, there is a complete analogy between the original Schrödinger equation (72) and the stochastic Eq. (68).

Therefore, equation (68) will be called: "the mass-independent non-stationary three-dimensional stochastic Schrödinger-Euler-Poisson equation" (or, in short, the "time-dependent stochastic Schrödinger-Euler-Poisson equation").

In contrast to the time-dependent Schrödinger equation (69), the non-stationary stochastic Schrödinger-Euler-Poisson equation (68) is applicable to the study of time-varying averaged states of ChWP of any scale.

2.8 The Stochastic Equation of Imaginary Self-Diffusion

If in the stochastic Eq. (68), we equate to zero the locally averaged potential energy (i.e., $\langle u(\vec{r}, t) \rangle = 0$), then it takes the form of the stochastic equation of imaginary self-diffusion (which is a particular case of the Fokker-Planck-Kolmogorov equation)

$$\frac{\partial\psi(\vec{r}, t)}{\partial t} = \mp i \frac{3}{2} D \nabla^2 \psi(\vec{r}, t) \quad (73)$$

with a complex self-diffusion coefficient $B = i \frac{3}{2} D$.

Thus, the proposed stochastic system model presupposes the possibility of studying various variants of the ChWP behavior, depending on its averaged characteristics.

2.9 The Problem of Factor 3

In the absence of the dependence of the wave function on time, i.e., for $\partial\psi(\vec{r}, t)/\partial t = 0$ and $\langle \varepsilon(\vec{r}, t_0) \rangle = \varepsilon$, Eq. (67) becomes the stationary stochastic Schrödinger-Poisson-Euler Eq. (41)

$$-\frac{3\eta_r^2}{4} \nabla^2 \psi(\vec{r}) + \langle u(\vec{r}) \rangle \psi(\vec{r}) = \varepsilon \psi(\vec{r}), \quad (73a)$$

which is to be expected.

However, on the other hand, one would expect that the equation of imaginary self-diffusion (73) should have the form

$$\frac{\partial\psi(\vec{r}, t)}{\partial t} = \mp i D \nabla^2 \psi(\vec{r}, t), \quad (73b)$$

since the complex self-diffusion coefficient was determined as $B = iD$ {see Ex. (A2.19) in Appendix 2}.

That is, if instead of factor 3 in the numerator of the first term in equation (73a), there were factor 2, then Eq. (41) {or (73a)} would have the preferable form

$$-\frac{\eta_r^2}{2} \nabla^2 \psi(\vec{r}) + \langle u(\vec{r}) \rangle \psi(\vec{r}) = \varepsilon \psi(\vec{r}). \quad (73c)$$

In this case, Eq. (68) would also be more concise

$$\pm i \frac{\eta_r^2}{2D} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\eta_r^2}{2} \nabla^2 \psi(\vec{r}, t) + \langle u(\vec{r}, t) \rangle \psi(\vec{r}, t), \quad (73d)$$

and for $\partial \psi(\vec{r}, t) / \partial t = 0$ and $\langle u(\vec{r}) \rangle = 0$, which becomes Eq. (73b).

The ambiguity of the situation lies in the fact that if the Euler - Lagrange equation

$$\frac{\partial L}{\partial \psi} + \frac{\partial^2}{\partial x^2} \left(\frac{\partial L}{\partial \psi_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial L}{\partial \psi_{yy}} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{\partial L}{\partial \psi_{zz}} \right) = 0, \quad (73e)$$

were used instead of the Euler - Poisson equation (35h), then it is easy to make sure that the substitution into this equation of the Lagrangian (35g)

$$L = -\frac{\eta_r^2}{2} \psi(x, y, z) \nabla^2 \psi(x, y, z) + \psi^2(x, y, z) [\langle u(x, y, z) \rangle - \langle \varepsilon(x, y, z) \rangle]$$

leads to the expected result, i.e., obtaining equation

$$-\frac{\eta_r^2}{2} \nabla^2 \psi(\vec{r}) + \langle u(\vec{r}) \rangle \psi(\vec{r}) = \langle \varepsilon(\vec{r}) \rangle \psi(\vec{r}). \quad (73f)$$

It would seem that using the Euler-Lagrange equation (73d) is a correct way because in this equation, the factor 3 disappears.

But if instead of the Euler-Poisson-Elsgolts equation (66), we use the Euler-Lagrange equation

$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \psi_t} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial L}{\partial \psi_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial L}{\partial \psi_{yy}} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{\partial L}{\partial \psi_{zz}} \right) = 0, \quad (73g)$$

then after substituting into it the Lagrangian (64)

$$L = -\frac{\eta_r^2}{2} \psi(\vec{r}, t) \nabla^2 \psi(\vec{r}, t) + [\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t) \rangle] \psi^2(\vec{r}, t) \pm i \frac{\eta_r^2}{D} \psi(\vec{r}, t) \frac{\partial \psi(\vec{r}, t)}{\partial t}$$

the following equation is obtained

$$-\frac{\eta_r^2}{2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial x^2} + \psi(\vec{r}, t) [\langle u(\vec{r}, t) \rangle - \langle \varepsilon(\vec{r}, t) \rangle] = 0, \quad (73h)$$

in which the first-time derivative of the wave function $\left(\frac{\partial \psi(\vec{r}, t)}{\partial t} \right)$ is absent, and this equation does not correspond to the time-dependent Schrödinger equation (70).

We do not know which formalism is more correct: Euler-Lagrange or Euler-Poisson. We can only state the facts:

- using the Euler-Lagrange formalism, the derivation of the time-independent Schrödinger equation is obtained, in which there is no multiplier (factor) 3. However, this formalism does not allow the derivation of the time-dependent Schrödinger equation;
- when using the Euler-Poisson formalism, the derivation of the time-independent Schrödinger equation and the time-dependent Schrödinger equation is obtained up to coefficients. But in this case, there is a factor 3, not the expected factor of 2.

Let's formulate the essence of the problem using the example of a 2-dimensional functional of general form

$$I[f] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x_0, x_1, f, f_0, f_1, f_{00}, f_{01}, f_{11}, \dots, f_{11\dots1}) dx_1 dx_2, \tag{73i}$$

where $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, $f_{iij} = \frac{\partial^3 f}{\partial x_i \partial x_i \partial x_j}$, ... ($i = 0,1; j = 0,1$).

The extremals of this function can satisfy the Euler-Lagrange equation [22, 23]

$$\begin{aligned} \frac{\partial L}{\partial f} - \frac{\partial}{\partial x_0} \left(\frac{\partial L}{\partial f_0} \right) - \frac{\partial}{\partial x_1} \left(\frac{\partial L}{\partial f_1} \right) + \frac{\partial^2}{\partial x_0^2} \left(\frac{\partial L}{\partial f_{00}} \right) + \frac{\partial^2}{\partial x_0 \partial x_1} \left(\frac{\partial L}{\partial f_{01}} \right) + \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial L}{\partial f_{11}} \right) - \dots + \\ + (-1)^n \frac{\partial^n}{\partial x_1^n} \left(\frac{\partial L}{\partial f_{11\dots1}} \right) = 0, \end{aligned} \tag{73j}$$

or the Euler-Poisson-Elsgolts equation [24]

$$\begin{aligned} \frac{\partial L}{\partial f} - \frac{\partial}{\partial x_0} \left\{ \frac{\partial L}{\partial f_0} \right\} - \frac{\partial}{\partial x_1} \left\{ \frac{\partial L}{\partial f_1} \right\} + \frac{\partial^2}{\partial x_0^2} \left\{ \frac{\partial L}{\partial f_{00}} \right\} + \frac{\partial^2}{\partial x_0 \partial x_1} \left\{ \frac{\partial L}{\partial f_{01}} \right\} + \frac{\partial^2}{\partial x_1^2} \left\{ \frac{\partial L}{\partial f_{11}} \right\} - \dots + \\ + (-1)^n \frac{\partial^n}{\partial x_1^n} \left\{ \frac{\partial L}{\partial f_{11\dots1}} \right\} = 0, \end{aligned} \tag{73k}$$

where

$$\begin{aligned} \frac{\partial}{\partial x_i} \left\{ \frac{\partial L}{\partial f_i} \right\} = \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial f_i} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f}{\partial x_i} + \frac{\partial}{\partial f_0} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_0}{\partial x_i} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_1}{\partial x_i} \\ + \frac{\partial}{\partial f_{00}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{00}}{\partial x_i} + \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{11}}{\partial x_i} + \frac{\partial}{\partial f_{10}} \left(\frac{\partial L}{\partial f_i} \right) \frac{\partial f_{10}}{\partial x_i} + \dots \end{aligned} \tag{73l}$$

is the first complete, partial derivative concerning x_i ($i = 0,1$).

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} \left\{ \frac{\partial L}{\partial f_{ii}} \right\} = \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial L}{\partial f_{ii}} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial}{\partial f_0} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_0}{\partial x_i^2} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_1}{\partial x_i^2} \\ + \frac{\partial}{\partial f_{00}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{00}}{\partial x_i^2} + \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{11}}{\partial x_i^2} + \frac{\partial}{\partial f_{01}} \left(\frac{\partial L}{\partial f_{ii}} \right) \frac{\partial^2 f_{01}}{\partial x_i^2} + \dots \end{aligned} \tag{73m}$$

is the second complete partial derivative concerning x_i ($i = 0,1$)

$$\frac{\partial}{\partial x_i \partial x_j} \left\{ \frac{\partial L}{\partial f_{ij}} \right\} = \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial L}{\partial f_{ij}} \right) + \frac{\partial}{\partial f} \left(\frac{\partial L}{\partial f_{ij}} \right) \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial}{\partial f_0} \left(\frac{\partial L}{\partial f_{ij}} \right) \frac{\partial^2 f_0}{\partial x_i \partial x_j} + \frac{\partial}{\partial f_1} \left(\frac{\partial L}{\partial f_{ij}} \right) \frac{\partial^2 f_1}{\partial x_i \partial x_j} + \frac{\partial}{\partial f_{00}} \left(\frac{\partial L}{\partial f_{ij}} \right) \frac{\partial^2 f_{00}}{\partial x_i \partial x_j} + \frac{\partial}{\partial f_{11}} \left(\frac{\partial L}{\partial f_{ij}} \right) \frac{\partial^2 f_{11}}{\partial x_i \partial x_j} + \frac{\partial}{\partial f_{10}} \left(\frac{\partial L}{\partial f_{ij}} \right) \frac{\partial^2 f_{10}}{\partial x_i \partial x_j} + \dots \quad (73n)$$

is second mixed complete partial derivatives concerning x_i & x_j ($i = 0,1; j = 0,1$).

For the author, the Euler-Poisson-Elsgolts formalism looks preferable since it leads to the derivation of both Schrödinger equations up to constant coefficients. However, the question (“the problem of factor 3”) remains open.

2.10 The Stochastic Quantum Operators

Let’s show how operators are obtained in mass-independent stochastic quantum mechanics (MSQM). To do this, let us consider the model of a chaotically wandering particle (ChWP) shown in Figure 1.

During the chaotic movement of a particle in the vicinity of the conditional center, it constantly changes the direction of its movement. Therefore, a particle at each moment has an angular momentum

$$\vec{L} = \vec{r} \times \vec{p}, \quad (74)$$

where \vec{r} is the radius vector from the conditional center to the particle (Figure 1); $\vec{p} = m \vec{v}$ is the instantaneous value and direction of the particle momentum vector.

Let’s divide both sides of the vector Ex. (74) by the value m ; as a result, we obtain the angular velocity vector

$$\vec{\omega} = \frac{\vec{L}}{m|\vec{r}|^2} = \frac{\vec{r} \times \vec{v}}{|\vec{r}|^2}. \quad (75)$$

We represent the vector equation (75) in the component form

$$\omega_x = \frac{1}{|\vec{r}|^2} (y v_z - z v_y), \quad \omega_y = \frac{1}{|\vec{r}|^2} (z v_x - x v_z), \quad \omega_z = \frac{1}{|\vec{r}|^2} (x v_y - y v_x). \quad (76)$$

Let’s average these components

$$\overline{\omega_x} = \frac{1}{|\vec{r}|^2} (y \overline{v_z} - z \overline{v_y}), \quad \overline{\omega_y} = \frac{1}{|\vec{r}|^2} (z \overline{v_x} - x \overline{v_z}), \quad \overline{\omega_z} = \frac{1}{|\vec{r}|^2} (x \overline{v_y} - y \overline{v_x}). \quad (77)$$

We use the coordinate representation of the averaged components of the velocity vector (A2.2) (see Appendix 2) for $n = 1$

$$\overline{v_x} = \int_{-\infty}^{+\infty} \psi(x) \left(\pm i \eta_x \frac{\partial}{\partial x} \right) \psi(x) dx = \left(\pm \frac{i \eta_x}{2} \frac{\partial}{\partial x} \right) \int_{-\infty}^{+\infty} \psi(x) \psi(x) dx, \quad (78)$$

$$\overline{v_y} = \int_{-\infty}^{+\infty} \psi(y) \left(\pm i\eta_y \frac{\partial}{\partial y} \right) \psi(y) dy = \left(\pm \frac{i\eta_x}{2} \frac{\partial}{\partial y} \right) \int_{-\infty}^{+\infty} \psi(y) \psi(y) dy, \quad (79)$$

$$\overline{v_z} = \int_{-\infty}^{+\infty} \psi(z) \left(\pm i\eta_z \frac{\partial}{\partial z} \right) \psi(z) dz = \left(\pm \frac{i\eta_x}{2} \frac{\partial}{\partial z} \right) \int_{-\infty}^{+\infty} \psi(z) \psi(z) dz. \quad (80)$$

Let's prove that these expressions are genuine using the example of Ex. (78).

First, let's show that the expression

$$\overline{v_x} = \int_{-\infty}^{+\infty} \psi(x) \left(\pm i\eta_x \frac{\partial}{\partial x} \right) \psi(x) dx = \int_{-\infty}^{+\infty} \pm \frac{i\eta_x}{2} \frac{\partial}{\partial x} [\psi(x)\psi(x)] dx, \quad (81)$$

is true since equality holds

$$\begin{aligned} -\frac{i\eta_x}{2} \frac{\partial}{\partial x} [\psi(x)\psi(x)] &= -\frac{i\eta_x}{2} \left[\frac{\partial\psi(x)}{\partial x} \psi(x) + \psi(x) \frac{\partial\psi(x)}{\partial x} \right] = -\frac{i\eta_x}{2} \left[2\psi(x) \frac{\partial\psi(x)}{\partial x} \right] \\ &= -i\eta_x \left[\psi(x) \frac{\partial\psi(x)}{\partial x} \right] = \psi(x) \left(-i\eta_x \frac{\partial}{\partial x} \right) \psi(x). \end{aligned}$$

Since the operations of integration and differentiation are commutative, we finally write

$$\overline{v_x} = \int_{-\infty}^{+\infty} \psi(x) \left(-i\eta_x \frac{\partial}{\partial x} \right) \psi(x) dx = \int_{-\infty}^{+\infty} -\frac{i\eta_x}{2} \frac{\partial}{\partial x} [\psi(x)\psi(x)] dx = \left(-\frac{i\eta_x}{2} \frac{\partial}{\partial x} \right) \int_{-\infty}^{+\infty} \psi(x)\psi(x) dx,$$

this is what it was required to prove (Q.E.D).

Take into account that, for example, in (78)

$$\int_{-\infty}^{+\infty} \psi(x)\psi(x) dx = \int_{-\infty}^{+\infty} \rho(x) dx = 1 \quad (82)$$

this means that a wandering particle randomly moves in one direction or the other so that the average values of the components of its velocity (78)–(80) are equal to zero (i.e., $\overline{v_x} = 0$, $\overline{v_y} = 0$ and $\overline{v_z} = 0$).

Therefore, identities (78)–(80) are equivalent to mass-independent stochastic operators of the components of the velocity vector

$$\hat{v}_x = \mp \frac{\eta_r}{i} \frac{\partial}{\partial x}, \quad \hat{v}_y = \mp \frac{\eta_r}{i} \frac{\partial}{\partial y}, \quad \hat{v}_z = \mp \frac{\eta_r}{i} \frac{\partial}{\partial z}, \quad (83)$$

here, it is taken into account that for the isotropic case $\eta_x = \eta_y = \eta_z = \eta_r$.

The mass-independent stochastic operators (83), for $\frac{\hbar}{m} = \eta_r$, correspond to the operators of the components of the QM momentum vector [26]

$$\hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}, \quad \hat{p}_y = \frac{\hbar}{i} \frac{\partial}{\partial y}, \quad \hat{p}_z = \frac{\hbar}{i} \frac{\partial}{\partial z}.$$

Substituting expressions (78)-(80) into expressions (77), taking into account (83), we obtain mass-independent stochastic operators of the components of the ChWP angular velocity vector.

Substituting (86)-(88) into (85), taking into account (90), we obtain mass-independent stochastic operators of the components of the ChWP angular velocity vector

$$\begin{aligned}\hat{\omega}_x &= \mp \frac{\eta_r}{i|\vec{r}|^2} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \\ \hat{\omega}_y &= \mp \frac{\eta_r}{i|\vec{r}|^2} \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \\ \hat{\omega}_z &= \mp \frac{\eta_r}{i|\vec{r}|^2} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),\end{aligned}\tag{84}$$

which correspond to the quantum mechanical operators of the components of the angular momentum vector [26]

$$\hat{L}_x = \frac{\hbar}{i} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \quad \hat{L}_y = \frac{\hbar}{i} \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \quad \hat{L}_z = \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).$$

In a spherical coordinate system, stochastic operators (84) have the form

$$\begin{aligned}\hat{\omega}_x &= \mp \frac{\eta_r}{i|\vec{r}|^2} \left(\sin \phi \frac{\partial}{\partial \theta} - \text{ctg} \theta \cos \phi \frac{\partial}{\partial \phi} \right), \\ \hat{\omega}_y &= \mp \frac{\eta_r}{i|\vec{r}|^2} \left(\cos \phi \frac{\partial}{\partial \theta} - \text{ctg} \theta \sin \phi \frac{\partial}{\partial \phi} \right), \\ \hat{\omega}_z &= \mp \frac{\eta_r}{i|\vec{r}|^2} \frac{\partial}{\partial \phi}.\end{aligned}\tag{85}$$

The stochastic mass-independent operator of the square of the modulus of the angular velocity of the ChWP is

$$\hat{\omega}^2 = \hat{\omega}_x^2 + \hat{\omega}_y^2 + \hat{\omega}_z^2 = -\frac{\eta_r^2}{|\vec{r}|^4} \nabla_{\theta,\phi}^2,$$

where

$$\nabla_{\theta,\phi}^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.\tag{86}$$

All mass-independent stochastic quantum operators, analogous to the QM operators, can be obtained similarly. Only in the MSQM, instead of the ratio \hbar/m , there is a scale parameter η_r (33) {or (A1.45) in Appendix 1}, therefore, the BSCM is suitable for describing stochastic processes of any scale.

Similarly, the mathematical apparatus of the entire mass-independent stochastic quantum mechanics (MSQM) can be built, which almost entirely coincides with the mathematical apparatus of the QM. However, MSQM is based on the principles of “ordinary” (classical) logic and is suitable for describing quantum systems and effects of any scale.

2.11 The Uncertainty Principle in MSQM

The uncertainty in the velocity of a chaotically wandering particle (ChWP) is determined by the standard deviation

$$\sqrt{\overline{v_x^2}} = \sqrt{\int_{-\infty}^{+\infty} \psi(x) \left(\pm i \eta_x \frac{\partial}{\partial x} \right)^2 \psi(x) dx} = i \eta_x \sqrt{\int_{-\infty}^{+\infty} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} dx}, \quad (87)$$

and the uncertainty in the particle coordinate is determined by the volatility

$$\sqrt{\overline{x^2}} = \sqrt{\int_{-\infty}^{+\infty} \psi(x) x^2 \psi(x) dx}. \quad (88)$$

The joint uncertainty in coordinate and momentum can be represented as

$$\sqrt{\overline{v_x^2 x^2}} = \sqrt{\int_{-\infty}^{+\infty} \psi(x) \left(\pm i \eta_x \frac{\partial}{\partial x} \right)^2 x^2 \psi(x) dx} = \sqrt{-\eta_x^2 \int_{-\infty}^{+\infty} \psi(x) \frac{\partial^2 x^2}{\partial x^2} \psi(x) dx} = \sqrt{2} i \eta_x \quad (89)$$

This uncertainty principle of the MSQM is equivalent to the Heisenberg's uncertainty principle $\Delta x \Delta p_x \geq 2\pi\hbar$.

3. Conclusions

The article proposes to bring attention to the fact that all the objects and subjects around us simultaneously strive for two goals equal in importance but mutually opposite (i. e., antisymmetric):

- to the maximum possible order, which is expressed in the “principle of least action”,
- to the maximum possible chaos, which is expressed in the “principle of maximum entropy”.

In other words, there is no completely deterministic or random entity in the reality around us. Everything is subject to the simultaneous striving for change and orderliness.

To simultaneously take into account both of these tendencies, this work proposes a unified “principle of the average efficiency extremum”, in which both competing concepts, “order” and “chaos”, coexist.

The article considers the averaged states of a particle (i.e., a compact solid body) of any size, which, under the influence of fluctuations in the environment and/or various long-range forces, continuously wanders (oscillates, displaces) in 3-dimensional space like a Brownian particle.

A constantly trembling (shifting, oscillating) body is represented as a chaotically wandering particle (ChWP) with a continuous trajectory of motion and volume. At the same time, the internal structure of the ChWP is not considered, and rotation and deformations of its shape are not taken into account.

These ChWP include the **centers of** a valence electron in a hydrogen-like atom, a vibrating atom in the crystal lattice, the trembling yolk in a chicken egg, a floating moth in the vicinity of a burning lamp, etc. (see the beginning of § 2.1).

All these stochastic systems are similar and obey the same laws, taking into account different types of friction coefficients and viscosity of the medium surrounding the ChWP, as well as the

duration of the average period of its behavior. For example, to average the chaotic flights of a bird in a cage, a week of constant observation is required; averaging the chaotic displacements of the galactic nucleus relative to the main line of its motion in outer space will require millions of years of research. However, the results of such observations may be similar despite the significant difference in the scale of these events.

For example, in §3.6 [21](arXiv:1702.01880), it is predicted theoretically that the possible averaged states of the vibrating nucleus of a biological cell are similar to discrete states of a 3-dimensional quantum mechanical oscillator (i.e., an elementary particle under identical conditions). If these microscopic quantum effects are confirmed experimentally, we can outline ways to solve the measurement problem in stochastic quantum mechanics.

Within the mass-independent stochastic quantum mechanics (MSQM) framework, the problem of studying “pure” states of pico-particles is proposed to be solved as follows. It is necessary to find (or simulate) a stochastic macroscopic system similar to the investigated pico-scopic system (i.e., a chaotically wandering pico-particle) and carry out experiments with the macroscopic system without exerting a tangible effect on it. Then, the results of measurements at the macro level are projected onto possible similar manifestations of the picoscopic system.

Within the MSQM, such an approach to the study of “pure” states of picoscopic and megascopic systems is possible since the philosophical foundations of this stochastic mechanics are rooted in antiquity and are based on the belief that all levels of the Universe are similar to each other. In this sense, MSQM is a universal theory for all levels of organization of chaotically oscillating (shifting, trembling, wandering, moving) matter.

As applied to pico-particles (i.e., particles of atomic and subatomic scale), the MSQM corresponds to the stochastic quantum mechanics (SQM) of Edward Nelson [1]. In this case, the MSQM Eq.s (41) and (68), derived in this article based on the “principle of averaged efficiency extremum” of the ChWP, coincided with the corresponding Schrödinger Eq.s (42) and (69) up to coefficients.

In other words, in the mass-independent stochastic quantum mechanics (MSQM), the “pure” wave function $\psi(x, t)$ is the extremal of the functional of the averaged “efficiency” of the ChWP, written in the coordinate representation.

Thus, the stochastic equations (37), (41), (67), (68), and (73) obtained in the article are the conditions for finding the extremals $\psi(x, t)$ of the functional of the globally averaged “efficiency” of a chaotically wandering particle (ChWP).

It is important to note that this stochastic function is balanced about the simultaneous striving of any stochastic system immediately to two mutually opposite (antisymmetric) goals:

- to “order” (that is, to determinism with the most minor energy losses), and
- to “chaos” (i.e., to the maximum entropy or the most significant uncertainty).

Stochastic Eq.s (41) and (68) have a number of the following advantages over the corresponding Schrödinger equations (42) and (69):

1. In the reasoning given to derive stochastic Eq.s (41) and (68), no restrictions were imposed on a chaotically wandering particle (ChWP), except for the total energy balances (40) and (48). ChWP is an ordinary particle with volume, the trajectory of movement, location, and momentum every moment. In other words, the derivation of the stochastic Schrödinger-Euler-Poisson Eq.s (41) and (68) was obtained based on “ordinary” (classical) logic using the theory of probability, the theory of generalized functions, and the calculus of variations.

In 95 years since the appearance of Schrödinger's equations in 1926, many researchers have proposed various methods of deriving them, relying on the axioms of many different interpretations of quantum mechanics. Still, no universally recognized result has been obtained.

The scientific community has not succeeded in developing logically consistent justifications for the QM axioms. One of the reasons for the general dissatisfaction was the lack of a “beautiful” derivation of the Schrödinger equations.

2. The reduced Planck's constant ($\hbar = 1.055 \times 10^{-34}$ J/Hz) limits the scope of the Schrödinger equations (42) and (69), and the entire QM as a whole, to the description of atomic and subatomic scale phenomena.

The fact is that the ratio \hbar/m , which is explicitly or latently present in the Schrödinger equations, only then turns out to be physically significant when the particle mass m is minimal (for example, it is believed that the electron rest mass $m_e = 9.109 \times 10^{-31}$ kg).

However, the field of application of the stochastic Schrödinger-Euler-Poisson equations (41) and (68) is not limited by anything.

To use Eq.s (41) and (68) to describe the averaged states of any of the above stochastic systems, it is necessary to estimate their scale parameter η_r (33). For this, it is necessary to determine the standard deviation σ_r and the autocorrelation interval $\tau_{r\text{ cor}}$ of a three-dimensional random process in which the corresponding particle participates based on sufficiently long observations of the geometric center of the ChWP.

As an example, §3.6 [21](arXiv:1702.01880) shows the possibility of using the mass-independent stationary stochastic Schrödinger-Euler-Poisson equation (41) to obtain quantum numbers characterizing the possible averaged states of a chaotically oscillating nucleus of a biological cell during the interphase period.

3. The stochastic equation (41) is also applicable to describe the averaged states of a chaotically moving geometric center of an electron in the vicinity of the nucleus of a hydrogen-like atom. If, as a result of statistical processing of indirect observations of the chaotic behavior of a valence electron in such an atom, it turns out that its scale parameter is

$$\eta_{er} = \frac{2\sigma_{er}^2}{\tau_{er\text{ cor}}} \approx \frac{\hbar}{m_e} = \frac{1,055 \times 10^{-34}}{9,1 \times 10^{-31}} \approx 0,116 \times 10^{-3} \frac{m^2}{s},$$

then Eq.s (41) and (42) for this case will be almost entirely equivalent. In this sense, the time-independent Schrödinger equation (42) can be regarded as a particular case of the stationary stochastic Schrödinger-Euler-Poisson equation (41).

4. In Schrödinger's equations (42) and (68), the mass of an elementary particle is present. However, this mass cannot be directly measured by macroscopic measuring instruments.

On the other hand, in the stochastic Schrödinger-Euler-Poisson equations (41) and (71), there is no particle mass. In this case, the standard deviation σ_r and the autocorrelation interval $\tau_{r\text{ cor}}$ of a three-dimensional random process, in which the ChWP is involved, can always be estimated based on the statistical processing of the results of sufficiently long observations of practically any stochastic system. Therefore, the stochastic Eq.s (37), (41), (67), (68), and (73) obtained in this article are universal.

5. It has been introduced into the minds of several generations of physicists that an electron is a point-particle (that is, an entity that has no volume). This happened mainly because a point charge,

according to Coulomb's law, must explode due to the infinite repulsive force of different parts of a tiny charged particle.

There were also other arguments, such as L.D. Landau believed that if the electron were not a point particle, then in solid collisions with other particles, such stresses would arise that it would inevitably disintegrate into parts, but this does not happen in practice.

At the same time, the idea that elementary particles are point particles leads to many problems (in particular, ultraviolet divergences). In addition, the ordinary mind will never agree with the lack of volume in an actual particle.

Therefore, attempts were always made to escape the point-like nature of elementary particles. For example, renormalization and regulation (in particular, vacuum polarization around a point charge and charge diffusion due to fluctuations) form the illusion of particle volume, and many string theorists have moved from point to linear objects in a multidimensional space with 6 compactly folded dimensions.

Another thing is that quantum mechanics (QM) was initially formulated in such a way that it operates with wave functions and not with particles. Therefore, for the QM methodology, whether an elementary particle is a point object or not does not matter. However, in several experiments at the time of registration, elementary particles manifest themselves as clearly local formations, rather than wave functions, diffuse throughout the Universe. The followers of Niels Bohr bypassed this problem with the help of the so-called instant "state reduction". Many experts do not like that this process must proceed at an infinite speed. Hence, the adherents of the many-worlds interpretation of Hugh Everett found it more acceptable to assume that it is not the wave function of a particle dispersed throughout the world that instantly collapses into a point. Still, the whole world instantly goes into one of the many possible states corresponding to the result of the experiment.

At the same time, according to many scientists (also modern), the concept of puffiness of elementary particles should be replaced by the idea of their "geometric center". Although the "geometric center" of any object is a mental construction, it does not contradict common sense, and the study of the dynamics of the "geometric center" of a complex object sometimes simplifies poorly formalized tasks.

In the proposed article, the geometric center of the ChWP moves along a chaotic trajectory even though the shape of the ChWP can change, oscillate, and rotate.

6. Suppose we assume that the velocity and coordinate of a chaotically wandering particle (ChWP) are entirely independent (i.e., uncorrelated) quantities. In that case, a Brownian (and not quantum) particle is obtained and described by the diffusion equation (a particular case of the Fokker-Planck-Kolmogorov equation). For such a particle, the dispersion of its location increases with time. This is a model of the Markov process.

However, suppose a colloidal particle turns out to be in a small closed space or in the field of action of a potential force. In that case, the Markov character of the process is violated, and in many cases, the averaged behavior of the ChWP becomes quantum (i.e., its averaged states are quantized).

The theory of dynamic chaos deals with such processes, where, on the one hand, completely deterministic systems of differential equations with specific external and internal parameters can lead to pseudo-chaotic trajectories diverging in the Lyapunov fashion. On the other hand, in dynamic chaos, order periodically manifests itself, i.e., discrete ordered and/or periodic configurations (patterns).

“True” quantum indeterminism, according to the author, is because the dominance of two essential factors determines the average state of a dynamical system: the “principle of least action” and “the principle of maximum entropy”, i.e., the parity between the «order» and «chaos».

Although the entropy of a distributed system is not mentioned in this article, it is still frequently present. Schrödinger-Euler-Poisson equations obtained in this article based on the principle of “principle of averaged efficiency extremum” lead to solutions (i.e., to the squares of the modulus of wave functions), which are simultaneously extremals of both the functional of the globally averaged efficiency and the Shannon entropy functional.

At the beginning of the method for studying a stochastic distributed system proposed in this article, we abstracted from chaos through averaging. However, the averaged order corresponded to the maximum entropy (uncertainty), which guarantees stable equilibrium of the considered distributed system.

7. MSQM predicts that many stationary random processes (in which ChWP are involved) can transition from one stationary state to another by absorbing or releasing a specific part of the total mechanical energy.

For example, this is easy to check when a moth constantly chaotically flies around a luminous lamp. You can record his chaotic movements with a video camera for a long time. If you then scroll through the video recording at high speed, then the moth will not be visible on the screen, but there will be a stable, blurry, dark spot, which reflects the PDF of the location of its geometric center. It should be expected that if the moth is not disturbed by anything, then the blurred spot will resemble a Gaussian PDF with the most significant darkening in the area of the center of the light bulb. However, if the moth is somehow energetically influenced, for example, by heat or ultrasound with a particular frequency, then its average behavior can abruptly change. In this case, the blurred spot can change the configuration to the average shape of a ring or figure-eight, etc.

Also, the geometric center of a flower, depending on the intensity of gusts of wind, can, on average, describe a straight segment, a circle, an ellipse, a figure-eight, or another Lissajous figure.

Similar 2-D and 3-D quantum effects appear in all ChWP of any scale. This contains the main idea of mass-independent stochastic quantum mechanics (MSQM): “Studying stochastic objects of the macrocosm using conventional (benchmark) methods, we simultaneously obtain information about all similar objects of the microcosm and objects of cosmic scale.

The approach proposed in this paper makes it possible to derive the equations of nonrelativistic mass-independent stochastic quantum mechanics (MSQM) (37), (41), (67), (68) and (73) based on principles fundamentally different from the ideological foundations of modern QM interpretations: Copenhagen, Many-worlds, Consistent histories, Decoherence, etc., but the mathematical apparatus of the MSQM turns out to be completely analogous to the mathematical apparatus of the QM.

Apparently, many other quantum field theory equations can be obtained similarly, such as the Klein-Fock-Gordon equation, the Dirac equation, the Maxwell equation, etc. The algorithm for deriving them may be similar to the approach given in this work:

- 1) the average energy balance condition of the stochastic system is recorded;
- 2) mass is extracted from the average energy balance condition, and the “efficiency” of this system is obtained;
- 3) to execute globally average the “efficiency” of the stochastic system under consideration;

- 4) all the averaged terms in the integrand of the globally averaged “efficiency” are represented through the PDF $\rho(x)$ and PDF $\rho(p_x)$;
- 5) all the terms of the Lagrangian of the globally averaged “efficiency” of the system are converted into a coordinate representation;
- 6) the equation for the extremals of the resulting function is determined through the calculus of variations (i.e., using the Euler - Poisson equation).

Further research may confirm this approach's validity in deriving the field theory equations.

We hope that this work will assist in discovering and studying quantum phenomena not only of the microcosm but also of the macro- and Worlds.

Appendix

Appendix 1

A1. Determination of the PDF of the Derivative of a Stationary and Pseudo-Stationary Differentiated Random Process

Consider several realizations of the random process $\xi(t)$ (see Figure A1.1).

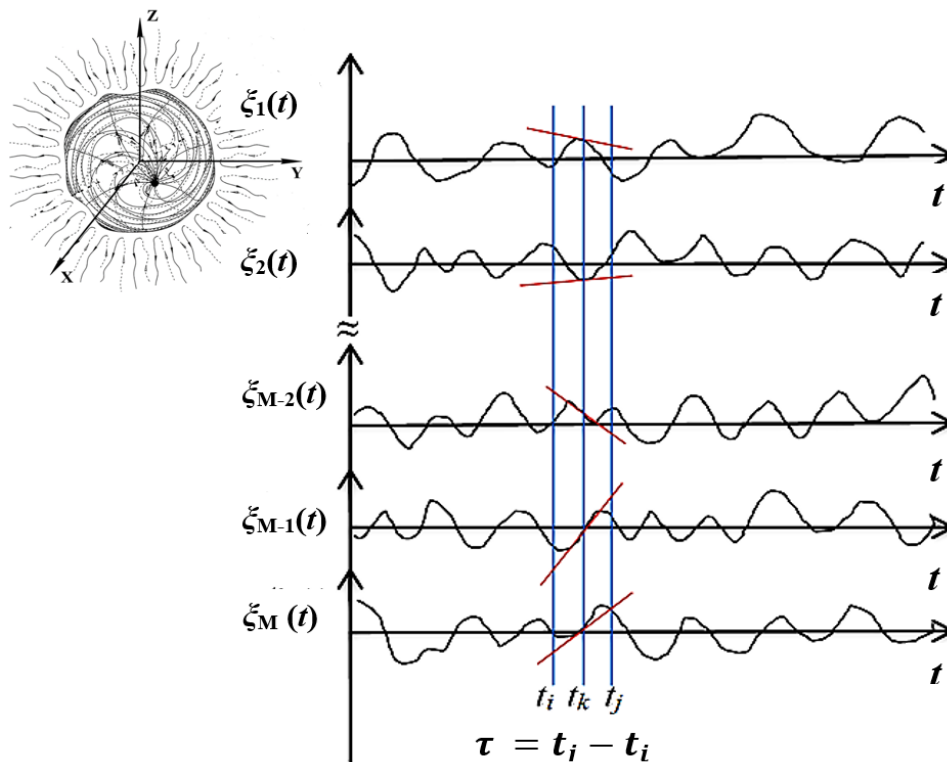


Figure A1.1 The realizations of a differentiable stationary or pseudo-stationary random process $\xi(t)$. These realizations can be interpreted, for example, as time changes in the projection of the location of a wandering particle on the X axis (see Figure 1), i.e., $x(t) = \xi(t)$.

In General, this process is non-stationary, but we assume that all the averaged characteristics of this process in the section t_i do not significantly differ from its similar averaged characteristics in the section t_j . That is, we require that all the moments and central moments of this process in the

section t_i are approximately equal to the corresponding moments and central moments in the section t_j when $\tau = t_j - t_i$ tending to zero. For example,

$$\overline{\xi(t_i)} \approx \overline{\xi(t_j)}; \tag{A1.1}$$

$$\overline{\xi^2(t_i)} \approx \overline{\xi^2(t_j)}, \text{ etc.} \tag{A1.2}$$

In other words, the considered random process $\xi(t)$ is either stationary or close to it. However, in each section t_m , all the averaged characteristics of such a process remain unchanged. We will call such a process a “pseudo-stationary random process” (PSRP) for convenience.

All conclusions about the PSRP made in this appendix also apply to the stationary random process (SRP).

There is a known procedure for obtaining the PDF $\rho(\xi'_k)$ of the derivative of a random process $\xi'(t) = d\xi(t)/dt$ with a known two-dimensional PDF of a random stationary process [27, 28]

$$\rho(\xi_i, \xi_j) = \rho(\xi_i, t_i; \xi_j, t_j). \tag{A1.3}$$

In Ex. (A1.3), we make the change of variables

$$\xi_i = \xi_k - \frac{\tau}{2} \xi'_k; \quad \xi_j = \xi_k + \frac{\tau}{2} \xi'_k; \quad t_i = t_k - \frac{\tau}{2}; \quad t_j = t_k + \frac{\tau}{2}, \tag{A1.4}$$

where

$$\tau = t_j - t_i; \quad t_k = \frac{t_j + t_i}{2}, \tag{A1.5}$$

with the Jacobian $[J] = \tau$.

As a result, from the two-dimensional PDF (A1.3), we obtain

$$\rho_2(\xi_k, \xi'_k) = \lim_{\tau \rightarrow 0} \tau \rho_2\left(\xi_k - \frac{\tau}{2} \xi'_k, t_k - \frac{\tau}{2}; \xi_k + \frac{\tau}{2} \xi'_k, t_k + \frac{\tau}{2}\right). \tag{A1.6}$$

Integrating (A1.6) over ξ_k , we find the required PDF $\rho(\xi'_k)$ in the section t_k [27]:

$$\rho(\xi'_k) = \int_{-\infty}^{\infty} \rho(\xi_k, \xi'_k) d\xi_k. \tag{A1.7}$$

Let’s now consider the possibility of obtaining the PDF $\rho(\xi'_k)$ for a known one-dimensional PDF $\rho(\xi)$.

To solve this problem, we use the following properties of random processes:

1. A two-dimensional PDF of a random process can be represented as [27, 28]

$$\rho(\xi_i, t_i; \xi_j, t_j) = \rho(\xi_i, t_i) \rho(\xi_j, t_j / \xi_i, t_i), \tag{A1.8}$$

where $\rho(\xi_j, t_j / \xi_i, t_i)$ is the conditional PDF.

2. For any PSRP and SRP, the approximate identity is valid

$$\rho(\xi_i, t_i) \approx \rho(\xi_j, t_j). \tag{A1.9}$$

3. The conditional PDF of a random process at $\tau = t_i - t_j$ tending to zero becomes the delta function

$$\lim_{\tau \rightarrow 0} \rho(\xi_j, t_j / \xi_i, t_i) = \delta(\xi_j - \xi_i). \tag{A1.10}$$

Using the above properties, we prepare a random process in the interval

$$[t_i = t_k - \tau/2; t_j = t_k + \tau/2] \text{ as } \tau \rightarrow 0,$$

based on the following procedure.

The PDF $\rho(\xi_i) = \rho(\xi_i, t_i)$ in the section t_i and the PDF $\rho(\xi_j) = \rho(\xi_j, t_j)$ in the section t_j can always be represented as a product of two functions

$$\rho(\xi_i) = \varphi(\xi_i)\varphi(\xi_i) = \varphi^2(\xi_i), \tag{A1.11}$$

$$\rho(\xi_j) = \varphi(\xi_j)\varphi(\xi_j) = \varphi^2(\xi_j),$$

where $\varphi(\xi_i)$ is the probability amplitude (PA) of the random variable ξ_i in the section t_i ; $\varphi(\xi_j)$ is a PA of a random variable ξ_j in the section t_j .

For PSRP, the approximate expression is valid

$$\varphi(\xi_i) \approx \varphi(\xi_j), \tag{A1.12}$$

which can be verified by taking the square root of both parts (A1.9).

For SRP, the approximate relation (A1.12) becomes the equality

$$\varphi(\xi_i) = \varphi(\xi_j). \tag{A1.12a}$$

Note that the approximate Ex. (A.1.12) at $\tau \rightarrow 0$ for the majority of non-stationary random processes (including for PSRP) also turns into the equality

$$\varphi(\xi_i, t_i) = \lim_{\tau \rightarrow 0} \varphi(\xi_j, t_j = t_i + \tau). \tag{A1.13}$$

When the condition (A.1.12) [or (A.1.12a)] is satisfied, Ex. (A1.8) can be represented in the following form

$$\rho(\xi_i, \xi_j) \approx \varphi(\xi_i)\rho(\xi_j/\xi_i)\varphi(\xi_j), \tag{A1.14}$$

where $\rho(\xi_j/\xi_i)$ is the conditional PDF.

Let's write (A.1.14) in expanded form

$$\begin{aligned} & \rho \left[\xi_i, t_i = t_k - \frac{\tau}{2}; \xi_j, t_j = t_k + \frac{\tau}{2} \right] \approx \\ & \approx \varphi \left[\xi_i, t_i = t_k - \frac{\tau}{2} \right] \rho \left[\xi_j, t_j = t_k + \frac{\tau}{2} / \xi_i, t_i = t_k - \frac{\tau}{2} \right] \varphi \left[\xi_j, t_j = t_k + \frac{\tau}{2} \right]. \end{aligned} \tag{A1.15}$$

Let τ tend to zero in (A1.15) so that the given time interval contracts uniformly on the left and right at the middle moment of time $t_k = (t_j + t_i)/2$. In this case, taking into account (A1.10) from (A1.14), we obtain the exact equality

$$\lim_{\tau \rightarrow 0} \rho(\xi_i, \xi_j) = \lim_{\tau \rightarrow 0} \{ \varphi(\xi_i) \rho(\xi_j/\xi_i) \varphi(\xi_j) \} = \varphi(\xi_{ik}) \delta(\xi_{jk} - \xi_{ik}) \varphi(\xi_{jk}), \quad (A1.16)$$

where ξ_{ik} is the result of the tendency of the random variable $\xi(t_i)$ to the random variable $\xi(t_k)$ on the left; ξ_{jk} is the result of the tendency of the random variable $\xi(t_j)$ to the random variable $\xi(t_k)$ from the right.

Integrating both sides of Ex. (A1.16) over ξ_{ik} and ξ_{jk} , we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi_{ik}) \delta(\xi_{jk} - \xi_{ik}) \varphi(\xi_{jk}) d\xi_{ik} d\xi_{jk} = 1. \quad (A1.17)$$

In (A1.17), the properties of the δ -function are considered.

Let's set the specific form of the δ -function. To do this, consider a random Markov process for which the diffusion equation (a particular case of the Fokker-Planck - Kolmogorov equation) is valid

$$\frac{\partial \rho(\xi_j/\xi_i)}{\partial t} = B \frac{\partial^2 \rho(\xi_j/\xi_i)}{\partial \xi^2}, \quad (A1.18)$$

where B is the diffusion coefficient.

One of the solutions of this differential equation, as is well known, has the form

$$\rho(\xi_j, t_j/\xi_i, t_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{iq(\xi_j - \xi_i) - q^2 B(t_j - t_i)\} dq, \quad (A1.19)$$

where q is the generalized frequency.

For $\tau = t_j - t_i \rightarrow 0$ from (A.1.19), we obtain one of the definitions of the δ -function

$$\lim_{\tau \rightarrow 0} \rho(\xi_j/\xi_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{iq(\xi_{jk} - \xi_{ik})\} dq = \delta(\xi_j - \xi_i). \quad (A1.20)$$

This result was obtained for the case as $\tau \rightarrow 0$. Therefore, the δ -function (A1.20) can correspond not only to a Markov random process but also to many other stationary and non-stationary random processes. In other words, one could immediately assume that the δ -function for the PSRP has the form (A.1.20) without referring to the equation (A.1.18).

Let's substitute δ -function (A.1.20) into Ex. (A.1.17)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi_{ik}) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{iq(\xi_{jk} - \xi_{ik})\} dq \right] \varphi(\xi_{jk}) d\xi_{ik} d\xi_{jk} = 1, \quad (A1.21)$$

and change the order of integration in (A1.21)

$$\int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi_{ik}) \exp\{-iq\xi_{ik}\} d\xi_{ik} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi_{jk}) \exp\{iq\xi_{jk}\} d\xi_{jk} \right] dq = 1. \quad (A1.22)$$

Let's take into account that, according to (A1.13), for the SRP and PSRP, the condition $\varphi(\xi_{ik}) = \varphi(\xi_{jk})$ is fulfilled. Therefore, Ex. (A.1.22) can be represented as

$$\int_{-\infty}^{\infty} \varphi(q)\varphi^*(q) dq = 1, \quad (A1.23)$$

where

$$w(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi_k) \exp\{-iq\xi_k\} d\xi_k, \quad (A1.24)$$

$$w^*(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi_k) \exp\{iq\xi_k\} d\xi_k. \quad (A1.25)$$

The integrand $\varphi(q)\varphi^*(q)$ in (A1.23) meets all the requirements of the PDF $\rho(q)$ of the random variable q :

$$\rho(q) = w(q)w^*(q) = |w(q)|^2. \quad (A1.26)$$

Let's clarify the physical meaning of q .

The features of the considered random process impose the following restrictions on the generalized frequency q :

- 1) the random variable q should characterize the random process $\xi(t)$ in the section t_k (see Figure A1.1), i.e., in the interval $\tau = t_j - t_i$ tending to 0;
- 2) the variable q must belong to the set of real numbers; it must take any value from the range $[-\infty, \infty]$.

These requirements are satisfied by the following random values associated with the PSRP (or SRP) in the time interval τ :

$$\xi'_i = \frac{\partial \xi_k}{\partial t}, \quad \xi''_i = \frac{\partial^2 \xi_k}{\partial t^2}, \quad \dots, \quad \xi_i^{(n)} = \frac{\partial^n \xi_i}{\partial t^n}. \quad (A1.27)$$

To clarify which values are associated with the generalized frequency q , consider one implementation of the process under study (see Figure A1.1). The function $\xi(t)$ in the interval at $\tau < \tau_{cor}$ [where τ_{cor} is the autocorrelation interval of the random process $\xi(t)$] can be expanded in the Maclaurin series

$$\xi(t_j) = \xi(t_i) + \xi'(t_i)\tau + \frac{\xi''(t_i)}{2}\tau^2 + \dots + \frac{\xi^{(n)}(t_i)}{n!}\tau^n + \dots \tag{A1.28}$$

The Ex. (A1.28) is presented in the following form

$$\frac{\xi_j - \xi_i}{\tau} = \xi'_i + \frac{\xi''_i}{2!}\tau + \dots + \frac{\xi^{(n)}_i \tau^{n-1}}{n!} + \dots \tag{A1.29}$$

where $\xi(t_i) = \xi_i$, $\xi(t_j) = \xi_j$, and we tend τ to zero.

In this case, (A1.29) is reduced to the expression

$$\lim_{\tau \rightarrow 0} \frac{\xi_j - \xi_i}{\tau} = \xi'_k, \tag{A1.30}$$

where $\xi_k = \xi(t_k)$ (see Figure A1.1).

Therefore, it remains to assume that the generalized frequency q in Ex.s (A1.23)-(A1.26) is linearly related only to ξ'_k , i.e.

$$q = \frac{\xi'_k}{\eta}, \tag{A1.31}$$

where η is the scale parameter.

The Ex. (A1.31) can be obtained in another way.

Each exponential, for example, from the integral (A1.24), corresponds to a harmonic function with frequency q

$$\exp\{-iq\xi(t)\} \rightarrow \xi_k(t) = A\sin(qt), \tag{A1.32}$$

this is one of the harmonic components of the random process $\xi(t)$.

Each frequency q , in turn, corresponds to the tangent of the angle of inclination of the tangent line to the harmonic function with a given frequency (see Figure A1.2), that is, $q \sim tg\alpha = \xi'(t)$.

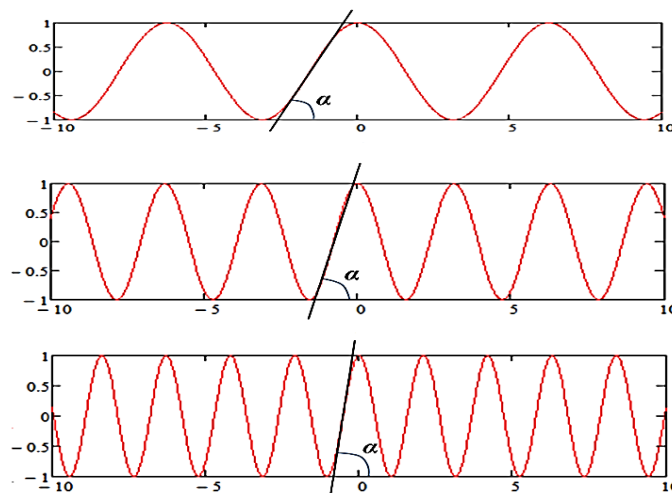


Figure A1.2 The more significance of the frequency q of the harmonic function, the greater the angle α between the tangent to this function and the t axis.

Differentiating Ex. (A1.32), we obtain $\xi'_k(t) = qA\cos(qt)$, whence it follows

$$q = \lim_{t \rightarrow 0} \frac{\xi'_k}{A\cos(qt)} = \frac{\xi'_k}{A}. \tag{A1.33}$$

For $A = \eta$ Ex.s (A1.31) and (A1.33) coincide.

Substituting (A1.31) into (A1.23)-(A1.26), we obtain the following required procedure for obtaining the PDF $\rho(\xi', t)$ of a pseudo-stationary random process (PSRP) or stationary random process (SRP) $\xi(t)$ in any section t_k for a known one-dimensional PDF $\rho(\xi, t)$ in the same section:

1. A given one-dimensional PDF $\rho(\xi, t)$ is represented as a product of two probability amplitudes (PA) $\varphi(\xi)$:

$$\rho(\xi, t) = \varphi(\xi, t)\varphi(\xi, t). \tag{A1.34}$$

2. Two Fourier transforms are performed

$$w(\xi', t) = \frac{1}{\sqrt{2\pi\eta}} \int_{-\infty}^{\infty} \varphi(\xi, t) \exp\{i\xi'\xi/\eta\} d\xi, \tag{A1.35}$$

$$w^*(\xi', t) = \frac{1}{\sqrt{2\pi\eta}} \int_{-\infty}^{\infty} \varphi(\xi, t) \exp\{-i\xi'\xi/\eta\} d\xi. \tag{A1.36}$$

3. Finally, for any section of the PSRP (or SRP), we obtain the required PDF of its derivative

$$\rho(\xi', t) = w(\xi', t)w^*(\xi', t) = |w(\xi', t)|^2. \tag{A1.37}$$

Once again, we note that the procedure (A1.34)–(A1.37) can be applied to any stationary and pseudo-stationary random processes {i.e., random processes with a slowly varying PDF $\rho(\xi, t)$ }, for which, as $\tau \rightarrow 0$, the δ -function takes the form (A1.20).

To clarify the physical meaning of the scale parameter η , consider a stationary random process $\xi(t)$ with a Gaussian distribution of the random variable ξ

$$\rho(\xi) = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left\{-\frac{(\xi - a_\xi)^2}{2\sigma_\xi^2}\right\}, \tag{A1.38}$$

where σ_ξ^2 and a_ξ are the variance and mathematical expectations of the process.

Performing the sequence of operations (A1.34)-(A1.37) with the PDF (A1.38), we obtain the PDF of the derivative of this random process

$$\rho(\xi') = \frac{1}{\sqrt{2\pi \left[\frac{\eta}{2\sigma_\xi}\right]^2}} \exp\left\{-\frac{\xi'^2}{2 \left[\frac{\eta}{2\sigma_\xi}\right]^2}\right\}. \tag{A1.39}$$

On the other hand, using the well-known procedure (A1.4)-(A1.7) for a similar case, we obtain [29]

$$\rho(\xi') = \frac{1}{\sqrt{2\pi\sigma_{\xi'}^2}} \exp\left\{-\xi'^2/2\sigma_{\xi'}^2\right\}, \tag{A1.40}$$

where $\sigma_{\xi'} = a_{\xi}/\tau_{\xi\text{ cor}}$,

here $\tau_{\xi\text{ cor}}$ is the autocorrelation interval of the initial random process $\xi(t)$.

Comparing the PDF (A1.39) and (A1.40), we find that

$$\eta = \frac{2\sigma_{\xi}^2}{\tau_{\xi\text{ cor}}}. \tag{A1.41}$$

The Ex. (A1.41) was obtained for a Gaussian random process, but the standard deviation σ_{ξ} and the autocorrelation interval $\tau_{\xi\text{ cor}}$ are the main characteristics of any SRP or PSRP. All other moments and central moments in the case of a non-Gaussian distribution of the random variable $\xi(t)$ will make an insignificant contribution to the Ex. (A1.41). Therefore, it can be argued with a high degree of reliability that Ex. (A1.41) applies to a large class of stationary and pseudo-stationary random processes.

In quantum mechanics, for the transition from the coordinate representation of the wave function of a pico-particle to its momentum representation, there is the procedure

$$\varphi(p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) \exp\{ip_x x/\hbar\} dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) \exp\{imx'x/\hbar\} dx, \tag{A1.42}$$

$$\varphi^*(p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) \exp\{-ip_x x/\hbar\} dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) \exp\{-imx'x/\hbar\} dx, \tag{A1.43}$$

where $\hbar = 1.055 \times 10^{-34}$ J/Hz is the reduced Planck's constant, and it is also taken into account that the x -component of the particle momentum p_x is related to its speed v_x (i.e., time derivative)

$$p_x = mv_x = m \frac{dx}{dt} = mx'. \tag{A1.44}$$

In the case when

$$\eta_x = \frac{2\sigma_x^2}{\tau_{x\text{ cor}}} = \frac{\hbar}{m} \text{ with dimension } (m^2/s), \tag{A1.45}$$

procedures (A1.34)-(A1.37) and (A1.42)-(A1.43) completely coincide.

From Ex. (A1.45), it follows that Planck's constant can be expressed through the main averaged parameters σ_x and $\tau_{x\text{ cor}}$ of a stationary (or pseudo-stationary) random process involving a randomly wandering pico-particle (for example, an electron).

At the same time, the field of application of the procedure (A1.42)–(A1.43) is limited by the smallness of the reduced Planck’s constant \hbar . The method (A1.34)–(A1.37) can be applied to any scale’s random and pseudo–stationary processes. Such random processes include chaotic oscillations of the center of mass of the nucleus of a biological cell, chaotic movements of the tip of a tree branch, chaotic changes in the position of the center of mass of the planet’s nucleus, etc.

Let’s note the following intermediate results:

1. For a stationary and pseudo-stationary random process $\xi(t) = x(t)$, the following procedure for obtaining the PDF $\rho(x')$ derivative of this process can be applied.

A given one-dimensional PDF $\rho(x)$ of a stationary process [or a slowly varying PDF $\rho(x, t)$ of a pseudo-stationary process] is represented as a product of two PA $\psi(x)$ or $\psi(x, t)$:

$$\rho(x) = \psi(x)\psi(x) \quad \text{or} \quad \rho(x, t) = \psi(x, t)\psi(x, t). \tag{A1.46}$$

a) For a stationary random process (SRP), two Fourier transforms are performed

$$\varphi(x') = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x) \exp\{ix'x/\eta_x\} dx = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x) \exp\{iv_x x/\eta_x\} dx, \tag{A1.47}$$

$$\varphi^*(x') = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x) \exp\{-ix'x/\eta_x\} dx = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x) \exp\{-iv_x x/\eta_x\} dx, \tag{A1.48}$$

and the desired PDF of the derivative of this process is determined

$$\rho(x') = \varphi(x') \varphi^*(x') = |\varphi(x')|^2, \tag{A1.49}$$

or

$$\rho(v_x) = \varphi(v_x) \varphi^*(v_x) = |\varphi(v_x)|^2, \tag{A1.50}$$

where

$$\eta_x = \frac{2\sigma_x^2}{\tau_{x\text{кор}}} = \frac{\hbar}{m} \tag{A1.51}$$

σ_x is the standard deviation of the initial stationary random process $x(t)$; $\tau_{x\text{кор}}$ is the autocorrelation interval of this process.

In § 2.6 of the article [30](arXiv:2007.13527), the procedure (A1.46)–(A1.50) is applied to obtain the PDF $\rho(x')$ of the derivative of stationary random processes with distribution laws: Gaussian, uniform, Laplace, Cauchy and sinusoidal.

b) For a pseudo-stationary random process (PSRP), two Fourier transforms are performed

$$\varphi(x', t) = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \psi(x, t) \exp\{ix'x/\eta_x(t)\} dx = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \psi(x, t) \exp\{iv_x x/\eta_x(t)\} dx, \tag{A1.52}$$

$$\varphi^*(x', t) = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \psi(x, t) \exp\{-ix'x/\eta_x(t)\} dx = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \psi(x, t) \exp\{-iv_x x/\eta_x(t)\} dx, \quad (A1.53)$$

and the required PDF of the derivative of this process is determined at each time moment t

$$\rho(x', t) = \varphi(x', t)\varphi^*(x', t) = |\varphi(x', t)|^2, \quad (A1.54)$$

or

$$\rho(v_x, t) = \varphi(v_x, t)\varphi^*(v_x, t) = |\varphi(v_x, t)|^2, \quad (A1.55)$$

where

$$\eta_x(t) = \frac{2\sigma_x^2(t)}{\tau_{x\text{ cor}}(t)} = \frac{\hbar}{m(t)}; \quad (A1.56)$$

$\sigma_x(t)$ is the standard deviation of the initial pseudo-stationary random process $x(t)$ from its mean value at time t ; $\tau_{x\text{ cor}}(t)$ is the autocorrelation interval of this process at time t .

2. The procedure (A1.46)-(A1.51) up to the proportionality coefficient η coincides with the quantum-mechanical procedure (A1.42)-(A1.43) of transition from the coordinate representation to the impulse one. But the quantum-mechanical procedure (A1.42)-(A1.43) was obtained using a rather unobvious (exotic) hypothesis about the possible existence of de Broglie's waves of matter (which were never discovered). In contrast, the procedure (A1.46)-(A1.51) is obtained based on a detailed analysis of a differentiable random process with the only assumption (which may be questioned) that the δ -function has the form (A1.20). In this regard, it is interesting to analyze which procedures for the transition from PDF $\rho(x)$ to PDF $\rho(x')$ can lead to other types of δ -function?

Also, there is no need to use Louis de Broglie's hypothesis of the existence of matter waves to describe the diffraction of particles by a crystal. In the article [30](arXiv:2007.13527), it is shown that, based on the laws of geometric optics and the theory of probability, a formula was obtained for calculating the volumetric scattering diagrams of particles on a multilayer periodic surface of a crystal.

3. In the case of studying the chaotic behavior of pico-particles, the ratio \hbar/m can be expressed through the main characteristics of the investigated random process (A1.45). In the author's opinion, this is a significant result since estimating the actual mass of a mobile elementary particle is almost impossible. Let's recall that in physical reference books, only the rest of the masses of elementary particles are given, which are determined indirectly on the basis of complex experiments. Whereas it is much easier to obtain an estimate of the standard deviation σ_x and the autocorrelation interval $\tau_{x\text{ cor}}$ of a randomly wandering particle. It is also essential that the reduced Planck constant \hbar loses its fundamental character and turns out to be the dimensional coefficient of proportionality between the particle mass and the ratio of the averaged characteristics of the random process.

Appendix 2

A2. Coordinate Representation of Characteristics of the Chaotically Wandering Particle (ChWP)

A2.1 Coordinate Representation of an Average Speed of the ChWP

For stationary and pseudo-stationary random processes (see Appendix 1), we prove the validity of equalities

$$\begin{aligned} \overline{x'^n} &= \overline{v_x^n} = \int_{-\infty}^{+\infty} \rho(x') x'^n dx'_x = \int_{-\infty}^{+\infty} \rho(v_x) v_x^n dv_x = \int_{-\infty}^{+\infty} \phi(v_x) v_x^n \phi(v_x) dv_x \\ &= \int_{-\infty}^{+\infty} \psi(x) \left(\pm i \eta_x \frac{\partial}{\partial x} \right)^n \psi(x) dx, \end{aligned} \tag{A2.1}$$

and

$$\begin{aligned} \overline{x'^n}(t) &= \overline{v_x^n}(t) = \int_{-\infty}^{+\infty} \rho(x', t) x'^n dx'_x = \int_{-\infty}^{+\infty} \rho(v_x, t) v_x^n dv_x \\ &= \int_{-\infty}^{+\infty} \phi(v_x, t) v_x^n \phi(v_x, t) dv_x = \int_{-\infty}^{+\infty} \psi(x, t) \left(\pm i \eta_x \frac{\partial}{\partial x} \right)^n \psi(x, t) dx, \end{aligned} \tag{A2.2}$$

where n is an integer, positive degree; η_x is the scale parameter (A1.51).

Experts in the field of QM are well aware of the proof of a similar expression

$$\overline{p_x^n} = \int_{-\infty}^{+\infty} \rho(p_x) p_x^n dp_x = \int_{-\infty}^{+\infty} \psi(p_x) p_x^n \psi(p_x) dp_x = \int_{-\infty}^{+\infty} \psi(x) \left(-i \hbar \frac{\partial}{\partial x} \right)^n \psi(x) dx,$$

see, for example, [31]. However, because of the importance of this proof for this article, we present it in a slightly modified form as applied to the features of mass-independent stochastic quantum mechanics (MSQM).

Let's use the Fourier transforms (A1.47) and (A1.48)

$$\phi(v_x) = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{i \frac{v_x x}{\eta_x}}}{(2\pi\eta_x)^{\frac{1}{2}}} dx = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{i \frac{v_x x}{\hbar}}}{(2\pi\hbar)^{\frac{1}{2}}} dx, \tag{A2.3}$$

$$\phi^*(v_x) = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{-i \frac{v_x x}{\eta_x}}}{(2\pi\eta_x)^{\frac{1}{2}}} dx = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{-i \frac{v_x x}{\hbar}}}{(2\pi\hbar)^{\frac{1}{2}}} dx. \tag{A2.4}$$

Substitute integrals (A2.3) and (A2.4) into the third part of Eq. (A2.1)

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \psi(x_i) \frac{e^{-i \frac{v_x x_i}{\eta_x}}}{(2\pi\eta_x)^{\frac{1}{2}}} dx_i v_x^n \int_{-\infty}^{+\infty} \psi(x_j) \frac{e^{i \frac{v_x x_j}{\eta_x}}}{(2\pi\eta_x)^{\frac{1}{2}}} dx_j \right] dv_x. \tag{A2.5}$$

It is easy to verify by direct verification that

$$v_x^n e^{i\frac{v_x x_j}{\eta_x}} = \left(-i\eta_x \frac{\partial}{\partial x_j}\right)^n e^{i\frac{v_x x_j}{\eta_x}}, \quad \text{or} \quad p_x^n e^{i\frac{p_x x_j}{\hbar}} = \left(-i\hbar \frac{\partial}{\partial x_j}\right)^n e^{i\frac{p_x x_j}{\hbar}}, \quad (\text{A2.6})$$

$$v_x^n e^{-i\frac{v_x x_i}{\eta_x}} = \left(i\eta_x \frac{\partial}{\partial x_i}\right)^n e^{-i\frac{v_x x_i}{\eta_x}}, \quad \text{or} \quad p_x^n e^{i\frac{p_x x_i}{\hbar}} = \left(i\hbar \frac{\partial}{\partial x_i}\right)^n e^{-i\frac{p_x x_i}{\hbar}}. \quad (\text{A2.6a})$$

Let's rewrite (A2.5) taking into account (A2.6)

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \psi(x_i) e^{-i\frac{v_x x_i}{\eta_x}} dx_i \int_{-\infty}^{+\infty} \psi(x_j) \left(-i\eta_x \frac{\partial}{\partial x_j}\right)^n e^{i\frac{v_x x_j}{\eta_x}} dx_j \right] dv_x.$$

or taking into account (A2.6a)

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \psi(x_i) \left(i\eta_x \frac{\partial}{\partial x_i}\right)^n e^{-i\frac{v_x x_i}{\eta_x}} dx_i \int_{-\infty}^{+\infty} \psi(x_j) e^{i\frac{v_x x_j}{\eta_x}} dx_j \right] dv_x. \quad (\text{A2.7a})$$

We integrate the second integral in the integrand (A2.7) n times by parts, and we will assume that $\psi(x)$ and its derivatives vanish at the integration boundaries $x = \pm\infty$. Performing these actions with expression (A2.7), we get [31]

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \psi(x_i) e^{-i\frac{v_x x_i}{\eta_x}} dx_i \int_{-\infty}^{+\infty} e^{i\frac{v_x x_j}{\eta_x}} \left(-i\eta_x \frac{\partial}{\partial x_j}\right)^n \psi(x_j) dx_j \right] dv_x, \quad (\text{A2.8})$$

or

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} \psi(x_i) e^{i\frac{v_x(x_j-x_i)}{\eta_x}} \left(-i\eta_x \frac{\partial}{\partial x_j}\right)^n \psi(x_j) dx_j \right] dv_x. \quad (\text{A2.9})$$

Similarly, we integrate the second integral in the integrand (A2.7a) n times by parts, and we will assume that $\psi(x)$ and its derivatives vanish at the integration boundaries $x = \pm\infty$. Performing these actions with expression (A2.7), we get

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} e^{-i\frac{v_x x_i}{\eta_x}} \left(i\eta_x \frac{\partial}{\partial x_i}\right)^n \psi(x_i) dx_i \int_{-\infty}^{+\infty} e^{i\frac{v_x x_j}{\eta_x}} \psi(x_j) dx_j \right] dv_x, \quad (\text{A2.8a})$$

or

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \psi(x_i) \left(i\eta_x \frac{\partial}{\partial x_i}\right)^n \psi(x_j) e^{i\frac{v_x(x_j-x_i)}{\eta_x}} dx_i \int_{-\infty}^{+\infty} dx_j \right] dv_x. \quad (\text{A2.9a})$$

Let's change the order of integration in (A2.9) and (A2.9a), i.e. first we will integrate over v_x

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx_j \psi(x_i) \left(-i\hbar \frac{\partial}{\partial x_j}\right)^n \psi(x_j) \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} e^{i\frac{v_x(x_j-x_i)}{\eta_x}} dv_x,$$

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} dx_j \int_{-\infty}^{+\infty} dx_i \psi(x_j) \left(i\hbar \frac{\partial}{\partial x_i} \right)^n \psi(x_i) \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} e^{i \frac{v_x(x_j-x_i)}{\eta_x}} dv_x.$$

There is a delta function in the expressions

$$\delta(x_j - x_i) = \frac{1}{2\pi\eta_x} \int_{-\infty}^{+\infty} e^{i \frac{v_x(x_j-x_i)}{\eta_x}} dv_x \quad \text{type} \quad \delta(x_j - x_i) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iq(x_j-x_i)} dq. \quad (\text{A2.10})$$

Therefore, we represent it in the form

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} \psi(x_i) \left(-i\eta_x \frac{\partial}{\partial x_j} \right)^n \psi(x_j) \delta(x_j - x_i) dx_j. \quad (\text{A2.11})$$

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} \psi(x_i) \left(i\eta_x \frac{\partial}{\partial x_j} \right)^n \psi(x_j) \delta(x_j - x_i) dx_j. \quad (\text{A2.11a})$$

Using the properties of the δ -function, we finally write

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} \psi(x) \left(\mp i\eta_x \frac{\partial}{\partial x} \right)^n \psi(x) dx, \quad (\text{A2.12})$$

$$\eta_x = \frac{2\sigma_x^2}{\tau_{x\text{кор}}} = \text{const.} \quad (\text{A2.12a})$$

thus, Ex. (A2.1) is proved for the case of a stationary random process (SSP).

For a pseudo-stationary random process (PSRP), Ex. (A2.2) is proved similarly. Performing operations similar to (A2.5)-(A2.15) using transformations (A1.52) and (A1.53), we obtain

$$\overline{v_x^n}(t) = \int_{-\infty}^{+\infty} \psi(x, t) \left(\mp i\eta_x \frac{\partial}{\partial x} \right)^n \psi(x, t) dx. \quad (\text{A2.13})$$

where

$$\eta_x = \frac{2\sigma_x^2}{\tau_{x\text{кор}}} \approx \eta_x(t) \approx \frac{2\sigma_x^2(t)}{\tau_{x\text{кор}}(t)} \approx \text{const.} \quad (\text{A2.13a})$$

In the general case, the scale parameter (A1.56) can change with time $\eta_x(t)$. However, in many non-stationary stochastic systems, it remains unchanged since the variance $\sigma_x^2(t)$ and the autocorrelation radius $\tau_{x\text{кор}}(t)$ of the pseudo-stationary random process (PSRP) change simultaneously and proportionally concerning each other. For example, a situation is possible when the variance of the pseudo-stationary random process (PSRP) changes with time according to the law $\sigma_x^2(t) = \sigma_x^2 \times (t - t_0)$, and its autocorrelation coefficient changes according to the same law $\tau_{x\text{кор}}(t) = \tau_{x\text{кор}} \times (t - t_0)$, then

$$\eta_x(t) = \frac{2\sigma_x^2(t)}{\tau_{xkop}(t)} \approx \frac{2\sigma_x^2 \times (t - t_0)}{\tau_{xkop} \times (t - t_0)} \approx \eta_x = \frac{2\sigma_x^2}{\tau_{xkop}} = const. \quad (A2.14)$$

The invariable ratio of the main averaged characteristics of the investigated random process will be called: “The law of proportional constancy of the scale parameter of the stochastic system.”

A2.2 Coordinate Representation of the Globally Averaged Change in the Mechanical Energiality of the ChWP

Let’s consider the case when a change in the probability amplitude $\psi(x, t)$ is associated with a change in the kinetic energetality of the ChWP.

Let’s return to the consideration of the conditional PDF (A1.19)

$$\rho(x_j, t_j / x_i, t_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{iq(x_j - x_i) - q^2 B(t_j - t_i)\} dq, \quad (A2.15)$$

where, according to (A1.31) and under condition (A2.14)

$$q = \frac{x'}{\eta_x} = \frac{v_x}{\eta_x}. \quad (A2.16)$$

If $\Delta x = x_j - x_i \rightarrow 0$, that from Ex. (A2.16) we obtain

$$\lim_{\Delta x \rightarrow 0} \rho(x_j, t_j / x_i, t_i) = \frac{1}{2\pi\eta_x} \int_{-\infty}^{\infty} \exp\left\{-\frac{v_x^2}{\eta_x^2} B(t_j - t_i)\right\} dv_x. \quad (A2.17)$$

Let’s take into account that

$$\frac{v_x^2}{2} = k_x, \quad (A2.18)$$

where k_x is the x -kinetic energetality of the particle, according to the terminology of mass-independent physics (14)-(16), i.e., physics freed from the heuristic concept of “mass”.

We also take into account that for some stochastic processes without friction, it should be assumed that the diffusion coefficient B^3 is a complex quantity

$$B = iD. \quad (A2.19)$$

³ The self-diffusion coefficient B is complex because the ChWP under consideration alternately diffuses in the chaotically fluctuating medium surrounding it, either in the forward direction or in the opposite direction in the vicinity of the conditional center of the stochastic system (see Figure 1). This vibrational behavior of the ChWP is due to the fact that it is alone and there is no pressure on it from the side of particles similar to it. Self-diffusion of ChWP occurs only due to the averaged excess (or lack) of chaotic exchange of kinetic energetality (i.e., the intensity of movement) between ChWP and the environment.

In the general case, $B = e^{-(\mu_0 - i\mu)} = D e^{i\mu} = D [\cos(\mu) + i \sin(\mu)]$, where μ_0 and μ are the parameters of complex self-diffusion, $D = e^{-\mu_0}$. In the particular case, for $\mu = \pi/2$, it turns out that $B = 0 + iD$. For $D = \eta_x$ the considered stochastic system turns out to be self-consistent.

Using expressions (A2.18) and (A2.19), we write equation (A2.17) in the following form

$$\lim_{\Delta x \rightarrow 0} \rho(x_j, t_j / x_i, t_i) = \frac{1}{2\pi\eta_x} \int_{-\infty}^{\infty} \exp\left\{-i \frac{v_x^2}{2} \frac{2D}{\eta_x^2} (t_j - t_i)\right\} dv_x. \quad (A2.20)$$

Substitute (A2.20) into an expression like Ex. (A.1.17)

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, t_i) \frac{1}{2\pi\eta_x} \exp\left\{-i \frac{v_x^2}{2} \frac{2D}{\eta_x^2} (t_j - t_i)\right\} \psi(x, t_j) dt_i dt_j \right] dv_x = 1. \quad (A2.21)$$

and change the order of integration

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, t_i) \frac{1}{2\pi\eta_x} \exp\left\{-i \frac{v_x^2}{2} \frac{2D}{\eta_x^2} (t_j - t_i)\right\} \psi(x, t_j) dt_i dt_j \right] dv_x = 1. \quad (A2.22)$$

We write Ex. (A2.22) similarly to Ex. (A1.22)

$$\int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x, t_j) \exp\left\{-i \frac{v_{xj}^2}{2} \frac{2D}{\eta_x^2} t_j\right\} dt_j \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x, t_i) \exp\left\{i \frac{v_{xi}^2}{2} \frac{2D}{\eta_x^2} t_i\right\} dt_i \right] dv_x = 1 \quad (A2.23)$$

From Ex. (A2.23), by analogy with (A1.22)-(A1.25), two Fourier transforms follow

$$\phi(k_{xj}, t_j) = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x, t_j) \exp\left\{-ik_{xj} \frac{2D}{\eta_x^2} t_j\right\} dt_j, \quad (A2.24)$$

$$\phi(k_{xi}, t_i) = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x, t_i) \exp\left\{ik_{xi} \frac{2D}{\eta_x^2} t_i\right\} dt_i. \quad (A2.25)$$

The change in the mechanical energy of the ChWP ε_{kx} due to the change in its kinetic energy at the point x at the intermediate time $t = (t_j + t_i)/2$ as $\tau = t_i - t_j \rightarrow 0$ (tending to zero) is on average equal to

$$\varepsilon_{kx} = \lim_{\tau \rightarrow 0} (k_{xj} + k_{xi})/2. \quad (A2.26)$$

Therefore, instead of integrals (A2.24) and (A2.25) for any intermediate time t , we can write

$$\phi(\varepsilon_{kx}, t) = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x, t) \exp\left\{-i \frac{D}{\eta_x^2} \varepsilon_{kx} t\right\} dt, \quad (A2.27)$$

$$\phi^*(\varepsilon_{kx}, t) = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \psi(x, t) \exp\left\{i \frac{D}{\eta_x^2} \varepsilon_{kx} t\right\} dt. \quad (A2.28)$$

According to expression (51), the globally averaged change {increase (+) or decrease (-)} of the mechanical energy of the ChWP is equal to

$$\pm \overline{\langle \varepsilon_k(x, t) \rangle} = \pm \int_{-\infty}^{\infty} \rho(\varepsilon_{kx}, t) \varepsilon_{kx} d\varepsilon_{kx} = \pm \int_{-\infty}^{\infty} \phi(\varepsilon_{kx}, t) \varepsilon_{kx} \phi^*(\varepsilon_{kx}, t) d\varepsilon_{kx}, \quad (A2.29)$$

where ε_{kx} is a local change in the mechanical energality of the ChWP due to a slight change in its kinetic energality in the direction of the X -axis; $\rho(\varepsilon_{kx}, t)$ is the probability distribution function (PDF) of changes ε_{kx} .

To write the Ex, (A2.29) in a coordinate representation, we perform actions similar to (A2.1)-(A2.13), taking into account the Fourier transformations (A2.27)-(A2.28) and the validity of the expressions

$$\varepsilon_{kx}^n e^{i\frac{2D}{\eta_x^2}\varepsilon_{kx}t} = \left(-i\frac{\eta_x^2}{D}\frac{\partial}{\partial t}\right)^n e^{i\frac{2D}{\eta_x^2}\varepsilon_{kx}t}, \quad (A2.30)$$

$$\varepsilon_{kx}^n e^{-i\frac{2D}{\eta_x^2}\varepsilon_{kx}t} = \left(i\frac{\eta_x^2}{D}\frac{\partial}{\partial t}\right)^n e^{-i\frac{2D}{\eta_x^2}\varepsilon_{kx}t}.$$

As a result, we obtain a coordinate representation of the globally averaged change in the mechanical energality of the ChWP due to the change in its averaged kinetic energality

$$\overline{\langle \varepsilon_k(x, t) \rangle} = \int_{-\infty}^{\infty} \rho(\varepsilon_{kx}, t) \varepsilon_{kx} d\varepsilon_{kx} = \pm i \frac{\eta_x^2}{D} \int_{-\infty}^{+\infty} \psi(x, t) \frac{\partial \psi(x, t)}{\partial t} dx. \quad (A2.31)$$

The proof of the validity of Ex. (A2.31) completely coincides with the proof of Ex. (A2.13)⁴.

Abbreviations and Definitions

- MSQM is the mass-independent stochastic quantum mechanics;
- QM is the quantum mechanics;
- PSRP is the pseudo-stationary random process;
- SQM is the stochastic quantum mechanics;
- SRP is the stationary random process;
- PA is the probability amplitude;
- PDF is the probability density function;
- ChWP is the chaotically wandering particle;
- Pico-particle is a particle or antiparticle with a size of $\sim 10^{-8} - 10^{-13}$ cm;
- Micro-particle is a particle with dimensions of $\sim 10^{-7} - 10^{-3}$ cm;
- Macro-particle is a compact bodies with dimensions of $\sim 10^{-2} - 10^4$ cm;
- $s = S/m$ is the “efficiency” of the particle with mass m ;
- $\varepsilon = E/m$ is the “mechanical energality” of the particle with mass m ;
- $u = U/m$ is the “potential energality” of the particle with mass m ;
- $k = T/m$ is the “kinetic energality” of the particle with mass m .

⁴ The similarity of the proofs of Ex.s (A2.13) and (A2.31) corresponds to the space-time symmetry between $p_x x$ and $E_x t$ (or $v_x x$ and $\varepsilon_x t$) in the de Broglie wave function $\psi = \exp\{-i(p_x x - E_x t)/\hbar\} = \exp\{-i(v_x x - \varepsilon_x t)m/\hbar\} = \exp\{-i(v_x x - \varepsilon_x t)\eta_x\}$, which underlies classical QM.

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Author Contributions

The author did all the research work for this study.

Competing Interests

The author has declared that no competing interests exist.

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